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# FINITE DIFFERENCE SOLUTION OF A NONLINEAR ELLIPTIC EQUATION*

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**Abstract**

The finite difference scheme is constructed for a nonlinear elliptic equation; its convergence is proved.

## I. The Scheme

Let $x=(x_1, x_2, x_3)\in R^3$ and

$$\Omega=\{x|\ 0<x_m<1,\ 1\leqslant m\leqslant 3\}.$$

The boundary of $\Omega$ is $\Gamma$. We consider the following problem

$$\begin{cases}-\nabla\cdot(\nu(U)\nabla U)=f, & x\in\Omega,\\ U=0, & x\in\Gamma,\end{cases}\tag{1}$$

where $f\in C^\alpha(\Omega+\Gamma)$, $0<\alpha<1$, and $\nu(\varphi)$ is a twice differentiable function. Assume that there are positive constants $\nu_0$, $\nu_1$, $C_1$ and $C_2$ such that for all $\varphi(x)$,

$$0<\nu_0\leqslant\nu(\varphi(x))\leqslant\nu_1,$$

$$|\nu'(\varphi(x))|\leqslant C_1,\quad |\nu''(\varphi(x))|\leqslant C_2.$$

Let $$a(V, W, \nu(\varphi))=\sum_{m=1}^{3}\int_\Omega \nu(\varphi(x))\nabla V(x)\cdot\nabla W(x)dx.$$

The generalized solution of (1) means such a function $U(x)\in H_0^1(\Omega)$ that

$$a(U, W, \nu(U))=\int_\Omega f(x)W(x)dx,\quad \forall W\in C_0^\infty(\Omega).\tag{2}$$

Douglas, Dupont[1] proved that the problem (1) possesses a unique generalized solution $U(x)\in C^{2+\alpha}(\Omega+\Gamma)$, and so $U(x)$ is the classical solution of (1) too. They also proposed a finite element scheme for solving (1) with the proof of convergence.

In this paper we construct a finite difference scheme for solving (1) and prove the existence of the approximate solution and the convergence.

Let $h$ be the mesh spacing of the variables $x_m(1\leqslant m\leqslant 3)$ and $Jh=1$, $J$ being an integer. Let $j_m(x)$ be an integer and

$$\Omega_h=\{x|\ x_m=j_m(x)h,\ 1<j_m(x)<J-1,\ 1\leqslant m\leqslant 3\}.$$

If $x\in\Omega_h$ and the distance from $x$ to $\Omega_h$ equals $h$, then we say $x\in\Gamma_h$.

Let $$e_1=(1,\ 0,\ 0),\ e_2=(0,\ 1,\ 0),\ e_3=(0,\ 0,\ 1).$$

We define $$u_{x_m}(x)=\frac{u(x+he_m)-u(x)}{h}.$$

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$$u_{\bar{x}_m}(x)=\frac{u(x)-u(x-he_m)}{h},$$

$$\Delta_{h,m}^{\nu(\varphi)}u(x)=\frac{1}{2}[(\nu(\varphi(x))u_{x_m}(x))_{\bar{x}_m}+(\nu(\varphi(x))u_{\bar{x}_m}(x))_{x_m}],$$

$$\Delta_h^{\nu(\varphi)}u(x)=\sum_{m=1}^{3}\Delta_{h,m}^{\nu(\varphi)}u(x).$$

Let $u_h(x)$ be the approximation of $U(x)$. The finite difference scheme for solving (1) is the following

$$\begin{cases}-\Delta_h^{\nu(u_h)}u_h(x)=f(x), & x\in\Omega_h,\\ u_h(x)=0, & x\in\Gamma_h.\end{cases}\tag{3}$$

Let $R_h(x)$ be the truncation error. Then

$$\begin{cases}-\Delta_h^{\nu(U)}U(x)=f(x)+R_h(x), & x\in\Omega_h,\\ U(x)=0, & x\in\Gamma_h.\end{cases}\tag{4}$$

Because $U(x)\in C^{3+\alpha}(\Omega+\Gamma)$, so $|R_h(x)|\to 0$, as $h\to 0$.

## II. Lemmas

We define

$$(u,v)=h^3\sum_{x\in\Omega_h}u(x)v(x),\quad \|u\|^2=(u,u),$$

$$\|u\|_{l^q}^q=h^3\sum_{x\in\Omega_h}|u(x)|^q,\quad \|u\|_\infty=\max_{x\in\Omega_h}|u(x)|,$$

$$|u|_1^2=\frac{1}{2}\sum_{m=1}^{3}(\|u_{x_m}\|^2+\|u_{\bar{x}_m}\|^2),$$

$$|u|_2^2=\frac{1}{2}\sum_{m=1}^{3}(|u_{x_m}|_1^2+|u_{\bar{x}_m}|_1^2),$$

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$$\|u\|_p^2=\sum_{r=1}^{p}|u|_r^2+\|u\|^2,$$

and

$$S_m(u,v,\nu(\varphi))=\frac{h}{2}\sum_{\substack{x\in\Gamma_h\\ x_m=0}}\nu(\varphi(x))u(x+he_m)v(x+he_m)$$
$$+\frac{h}{2}\sum_{\substack{x\in\Gamma_h\\ x_m=1}}\nu(\varphi(x))u(x-he_m)v(x-he_m),$$

$$S(u,v,\nu(\varphi))=\sum_{m=1}^{3}S_m(u,v,\nu(\varphi)).$$

Let $\mathscr{H}$ be the space of the mesh function $u(x)$ such that

$$u(x)=0,\quad\text{for}\quad x\in\Gamma_h.$$

We define the scalar product

$$(u,v)_{\mathscr{H}}=\frac{1}{2}\sum_{m=1}^{3}[(u_{x_m},v_{x_m})+(u_{\bar{x}_m},v_{\bar{x}_m})]+S(u,v,1)$$

and the norm

$$\|u\|_{\mathscr{H}}^2=(u,u)_{\mathscr{H}}.$$

Let

$$a_h(u,v,\nu(\varphi))=\frac{1}{2}\sum_{m=1}^{3}(\nu(\varphi),\ u_{x_m}v_{x_m}+u_{\bar{x}_m}v_{\bar{x}_m})+S(u,v,\nu(\varphi)).$$

**Lemma 1.** *For all $\varphi(x)$ and $u(x)$, we have*