THE QUASI-NEWTON METHOD IN PARALLEL CIRCULAR ITERATION*1)

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Newton iteration

$$x^{(k+1)}=x^{(k)}-\frac{f(x^{(k)})}{f'(x^{(k)})},\quad k=0,1,\cdots \tag{1}$$

is a most basic iteration for solving the numerical equation $f(x)=0$. If $f(x)$ is a monic polynomial of degree $n(n>1)$ with complex coefficients:

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n=\prod_{i=1}^{n}(x-\xi_i) \tag{2}$$

and its zeros $\xi_1,\cdots,\xi_n$ are different each other, we have

$$f(x)\approx\prod_{i=1}^{n}(x-x_i^{(k)}),\quad f'(x_i^{(k)})\approx\prod_{\substack{j=1\\j\neq i}}^{n}(x_i^{(k)}-x_j^{(k)}) \tag{3}$$

for some approximation $x_1^{(k)},\cdots,x_n^{(k)}$ of $\xi_1,\cdots,\xi_n$. Then the further approximation is

$$x_i^{(k+1)}=x_i^{(k)}-\frac{f(x_i^{(k)})}{\prod_{\substack{j=1\\j\neq i}}^{n}(x_i^{(k)}-x_j^{(k)})},\quad i=1,\cdots,n;\ k=0,1,\cdots. \tag{4}$$

This is just the parallel iteration proposed by Durand[1] and Kerner[2]. We see that this is a Newton method, which aims at the concrete task for finding all zeros of polynomial and applied approximation (3).

Let $W_i^{(k)}=[x_i^{(k)};r_i^{(k)}]$ denote disks in complex plane $\mathscr{C}$ with center $x_i^{(k)}$ and radius $r_i^{(k)}$

$$\{x\in\mathscr{C}:|x-x_i^{(k)}|\leqslant r_i^{(k)}\}.$$

Then under the operation rule of circular arithmetic[2)]

$$[x_1;r_1]\pm[x_2;r_2]=[x_1\pm x_2;r_1+r_2],$$

$$[x_1;r_1]\cdot[x_2;r_2]=[x_1x_2;|x_1|r_2+|x_2|r_1+r_1r_2],$$

$$\frac{1}{[x_2;r_2]}=\frac{1}{|x_2|^2-r_2^2}[\bar{x}_2;r_2],$$

$$\frac{[x_1;r_1]}{[x_2;r_2]}=[x_1;r_1]\cdot\frac{1}{[x_2;r_2]},$$

$0\overline{\in}[x_2;r_2]$, $\bar{x}_2$ denotes conjugate complex of x_2, an analogy of iteration (4) is

$$W_i^{(k+1)}=x_i^{(k)}-f(x_i^{(k)})\prod_{\substack{j=1\\j\neq i}}^{n}\frac{1}{x_i^{(k)}-W_j^{(k)}},\quad i=1,\cdots,n;\ k=0,1,\cdots, \tag{5}$$

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2) Any complex is regarded as a disk with radius 0.

which is called the quasi-Newton method in parallel circular iteration. In comparison with the parallel circular iterations proposed by Braess, Hadeler[3], Petković[4], Gargantini, Henrici[5] etc., the construction of iteration (5) is simpler.

According to the inclusion monotone of circular arithmetic, if $W_1^{(k)}, \cdots, W_n^{(k)}$ are isolate and contain $\xi_1, \cdots, \xi_n$ respectively, then $W_1^{(k+1)}, \cdots, W_n^{(k+1)}$ contain also $\xi_1, \cdots, \xi_n$ respectively. Therefore, the circular iteration is feasible if the isolation of $W_1^{(k+1)}, \cdots, W_n^{(k+1)}$ is guaranteed, and the rate of convergence may be described by the rate of

$$r^{(k)}=\max_{1\leq i\leq n} r_i^{(k)} \tag{6}$$

tending to 0. We may denote the isolation of disks $W_1^{(k)}, \cdots, W_n^{(k)}$ by

$$\vartheta^{(k)}=r^{(k)}/\rho^{(k)}, \tag{7}$$

where

$$\rho^{(k)}=\min_{\substack{1\leq i,j\leq n\\ j\neq i}} \min_{z\in W_j^{(k)}} |z-x_i^{(k)}|. \tag{8}$$

If $r^{(k)}\to 0$ $(k\to\infty)$, then

$$\rho^{(k)}\to \min_{\substack{1\leq i,j\leq n\\ j\neq i}} |\xi_j-\xi_i|\neq 0, \quad k\to\infty$$

and

$$\vartheta^{(k)}\asymp r^{(k)}, \quad k\to\infty. \tag{9}$$

Hence, the rate tended to 0 of $\vartheta^{(k)}$ represents directly the rate of convergence of circular iteration. We have the following theorem on $\vartheta^{(k)}$ for circular iteration (5).

Theorem. *Suppose that the initial disks $W_1^{(0)}, \cdots, W_n^{(0)}$ include the roots $\xi_1, \cdots, \xi_n$ of equation* (2) *respectively, and*

$$\vartheta^{(0)}\leq\frac{1}{3(n-1)}. \tag{10}$$

Then the sequences $\{W_i^{(k)}\}_{k=0}^{\infty}$ $(i=1, \cdots, n)$ produced by (5) *satisfy*

$$\vartheta^{(k+1)}\leq 3(n-1)(\vartheta^{(k)})^2, \quad k=0, 1, \cdots \tag{11}$$

and contract to roots $\xi_1, \cdots, \xi_n$ of equation (2) *respectively:*

$$\bigcap_{k=0}^{\infty} W_i^{(k)}=\xi_i, \quad i=1, \cdots, n. \tag{12}$$

Proof. From (5) and (2), we have

$$W_i^{(k+1)}-x_i^{(k)}=(x_i^{(k)}-\xi_i)\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{x_i^{(k)}-\xi_j}{x_i^{(k)}-W_j^{(k)}}, \quad i=1, \cdots, n;\ k=0, 1, \cdots. \tag{13}$$

Let

$$x=\mathrm{mid}[x;\, r], \quad r=\mathrm{rad}[x;\, r]. \tag{14}$$

By circular arithmetic we know that

$$\mathrm{mid}\,\frac{x_i^{(k)}-\xi_j}{x_i^{(k)}-W_j^{(k)}}=\frac{(x_i^{(k)}-\xi_j)(\overline{x}_i^{(k)}-\overline{x}_j^{(k)})}{|x_i^{(k)}-x_j^{(k)}|^2-(r_j^{(k)})^2},$$

$$\mathrm{rad}\,\frac{x_i^{(k)}-\xi_j}{x_i^{(k)}-W_j^{(k)}}=\frac{|x_i^{(k)}-\xi_j|r_j^{(k)}}{|x_i^{(k)}-x_j^{(k)}|^2-(r_j^{(k)})^2}, \quad i, j=1, \cdots, n;\ j\neq i;\ k=0, 1, \cdots.$$

Clearly,

$$|x_i^{(k)}-\xi_j|\leq|x_i^{(k)}-x_j^{(k)}|+r_j^{(k)},$$

$$\rho^{(k)}\leq|x_i^{(k)}-x_j^{(k)}|-r_j^{(k)}, \quad i, j=1, \cdots, n;\ j\neq i;\ k=0, 1, \cdots.$$

Thus,