BOUNDS ON CONDITION NUMBER OF A MATRIX*

HUANG HONG-CI (黄鸿慈)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

For each vector norm $\|x\|$, a matrix A has its operator norm $\|A\|=\max\limits_{x\neq0}\frac{\|Ax\|}{\|x\|}$ and a condition number $P(A)=\|A\|\|A^{-1}\|$. Let U be the set of the whole of norms defined on C^n. It is shown that for a nonsingular matrix $A\in C^{n\times n}$, there is no finite upper bound of $P(A)$ while $\|\cdot\|$ varies on U if $A\neq\alpha I$; on the other hand, it is shown that $\inf\limits_{\|\cdot\|\in U}\|A\|\|A^{-1}\|=\rho(A)\rho(A^{-1})$ and in which case this infimum can or cannot be attained, where $\rho(A)$ denotes the spectral radius of A.

Let $\|x\|$ be a norm defined on the linear space C^n. Then a matrix $A\in C^{n\times n}$, treated as a linear operator on C^n, has a norm $\|A\|=\max\limits_{x\neq0}\frac{\|Ax\|}{\|x\|}$ correspondingly. We denote by $P(A)=\|A\|\|A^{-1}\|$ the condition number of a nonsingular matrix A. This is a basic concept in numerical algebra and is important in some other fields of numerical analysis. Under certain circumstances, one takes the product of spectral radius $\rho(A)\rho(A^{-1})$, namely the ratio $|\lambda_1|/|\lambda_n|$ where λ_1 and λ_n are the largest and smallest eigenvalues of A by norm, to characterize the condition of A. $P(A)$ depends on the selected norm $\|\cdot\|$ while $\rho(A)\rho(A^{-1})$ is determined only by the matrix itself. Now we reveal their relationship.

Denote by U the set of the whole of norms defined on C^n. We begin with the upper bound of $P(A)$ while $\|\cdot\|$ varies on U. Obviously, when $A=\alpha I$, where I is the identity matrix and α is a nonzero scalar, $P(A)=1$ for any norm. Otherwise, we have

Theorem 1. *Let $A\in C^{n\times n}$, be nonsingular and $A\neq\alpha I$. Then there is no finite upper bound of $P(A)$ while $\|\cdot\|$ varies on U.*

Proof. Let $\|x\|_\infty=\max\limits_i|x_i|$, it is known that the corresponding norm of the matrix is

$$\|A\|_\infty=\max_i\sum_j|a_{ij}|. \tag{1}$$

Now we divide the matrices involved in the condition of the theorem into two cases: (1) at least one nonzero element on the off-diagonal, (2) a diagonal form with $a_{ii}\neq a_{jj}$ for some $i\neq j$ due to $A\neq\alpha I$.

In case (1), supposing $a_{ij}\neq0$, we take

$$Q_s=\begin{bmatrix}1&&&&\\&\ddots&&&\\&&s&&\\&&&\ddots&\\&&&&1\end{bmatrix}\text{(row } i\text{)},$$

a matrix different from the identity matrix only by the element $[Q_s]_{ii}=s\neq0$. With

* Received March 25, 1984.

notice of the nonsingularity of Q_s, we can define a norm $\|\cdot\|_{q(s)}$ with a parameter s such as

$$\|x\|_{q(s)} = \|Q_s x\|_\infty$$

and then correspondingly,

$$\|A\|_{q(s)} = \max_{x\neq 0} \frac{\|Ax\|_{q(s)}}{\|x\|_{q(s)}} = \max_{x\neq 0} \frac{\|Q_s Ax\|_\infty}{\|Q_s x\|_\infty} = \max_{y\neq 0} \frac{\|Q_s A Q_s^{-1} y\|_\infty}{\|y\|_\infty} = \|Q_s A Q_s^{-1}\|_\infty.$$

Through calculation and from (1) we can obtain

$$\|A\|_{q(s)} \geqslant |s a_{ij}|. \tag{2}$$

From (2), it can be seen that no finite upper bound of $\|A\|_{q(s)}$ exists while $|s|$ tends to infinity.

In case (2), supposing $a_{ii}\neq a_{jj}$, we take

$$T_\varepsilon = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 1+\varepsilon & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \text{(row } i) \\ \\ \text{(row } j), \\ \\ \\ \end{matrix}$$

where ε is a positive real number. We define

$$\|x\|_{t(\varepsilon)} = \|T_\varepsilon x\|_\infty$$

with a parameter ε. Through calculation we have

$$\|A\|_{t(\varepsilon)} = \|T_\varepsilon A T_\varepsilon^{-1}\|_\infty \geqslant \frac{1}{\varepsilon} |a_{ii} - a_{jj}|. \tag{3}$$

When $\varepsilon\to 0$, there is no finite upper bound of $\|A\|_{t(\varepsilon)}$. With notice of $\|A^{-1}\|\geqslant\rho(A^{-1})>0$ we conclude that $P(A)$ has no finite upper bound in both cases.

For any norm it is known that $\|A\|\,\|A^{-1}\|\geqslant\rho(A)\rho(A^{-1})$. Now we go further to prove the following theorem.

Theorem 2. *Let $A\in C^{n\times n}$, and be nonsingular. Then*

$$\inf_{\|\cdot\|\in U} \|A\|\,\|A^{-1}\| = \rho(A)\rho(A^{-1}). \tag{4}$$

Proof. Let Q be the matrix transforming A into Jordan canonical form, namely,

$$Q^{-1}AQ = J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{bmatrix}, \tag{5}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \tag{5a}$$

and is of order n_i, $\sum_{i=1}^{m} n_i = n$,

At the same time we have,