Vol. 2 No. 4. JOURNAL OF COMPUTATIONAL MATHEMATICS October 1984

INTERPOLATION FORMULAS OF INTERMEDIATE QUOTIENTS FOR DISCRETE FUNCTIONS WITH SEVERAL INDICES*

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§ 1

The imbedding relations and the interpolation formulas of Sobolev spaces are very important and powerful tools in the study of the linear and nonlinear theory of partial differential equations and systems. The finite difference method is commonly used in pratical and theoretical study of various problems of partial differential equations and systems of different type. Many authors are attracted to establish the imbedding relations and the interpolation formulas for the discrete functions. These are undoubtedly important in pratical and theoretical use of finite difference method. As in [1] the Sobolev inequalities are given for the discrete function on number axis. In [2] the difference imbedding theorems of the Sobolev space with weight are considered. In [3—5] for the discrete functions on finite segment the interpolation formulas are established. These interpolation formulas can be used to establish the convergence and stability of the finite difference schemes for the various problems of the nonlinear partial differential systems of different types. And these can also be used to construct the generalized and weak global solutions for the nonlinear systems of partial differential equations[3-9]. Now in the present note we want to consider the interpolation formulas of intermediate quotients for the discrete functions with several indices.

§ 2

Suppose that the finite interval $[0, l]$ is divided into the small segment grids by the points $x_j=jh$ $(j=0, 1, \cdots, J)$, where $Jh=l$, J is an integer and h is the stepsize. The discrete function $u_h=\{u_j\}$ $(j=0, 1, \cdots, J)$ is defined on the grid points x_j $(j=0, 1, \cdots, J)$. Let us denote $\Delta_+u_j=u_{j+1}-u_j$, $\Delta_-u_j=u_j-u_{j-1}$. For the norms of the discrete function $u_h=\{u_j\}$ $(j=0, 1, \cdots, J)$ and its difference quotients

$$\delta^k u_h=\left\{\frac{\Delta_+^k u_j}{h^k}\right\} \quad (j=0, 1, \cdots, J-k)$$

of order $k\geqslant 0$, we take the notations

$$\begin{aligned} \|\delta^k u_h\|_p&=\left(\sum_{j=0}^{J-k}\left|\frac{\Delta_+^k u_j}{h^k}\right|^p h\right)^{\frac{1}{p}}, \quad 1\leqslant p<\infty; \\ \|\delta^k u_h\|_\infty&=\max_{j=0,1,\cdots,J-k}\left|\frac{\Delta_+^k u_j}{h^k}\right|, \quad k\geqslant 0. \end{aligned} \tag{1}$$

* Received April 12, 1984

The interpolation formulas of the intermediate quotients for the discrete functions defined on finite interval $[0, l]$ can be stated as follows[3-5].

Theorem 1. *For any discret function $u_h=\{u_j\}$ $(j=0, 1, \cdots, J)$ defined on the finite interval $[0, l]$, there is the interpolation formula*

$$\|\delta^k u_h\|_p \leqslant K_1 \|u_h\|_2^{1-\frac{k+\frac{1}{2}-\frac{1}{p}}{n}} \left\{\|\delta^n u_h\|_2 + \frac{\|u_h\|_2}{l^n}\right\}^{\frac{k+\frac{1}{2}-\frac{1}{p}}{n}}, \tag{2}$$

where $2\leqslant p\leqslant\infty$, $0\leqslant k<n$ and K_1 is a constant independent of h, l and u_h.

§ 3

Let us introduce some notations for discrete functions with several indices.

Take a m-dimensional rectangular domain $Q^m=\{x \mid x=(x_1, \cdots, x_m) \mid 0\leqslant x_i\leqslant l_i, i=1, 2, \cdots, m\}$ in the m-dimensional Euclidean space $\mathbf{R}^m$, where $m\geqslant 1$ and $l_i>0$, $i=1, 2, \cdots, m$. For the rectangular domain Q^m we define $l=\min\limits_{i=1,2,\cdots,m}\{l_i\}>0$. Divide the rectangular domain Q^m into small grids by the parallel hyperplanes $x_i=x_{ij_i}$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$ where $x_{ij_i}=j_i h_i$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$ and $J_i h_i=l_i$ $(i=1, 2, \cdots, m)$. The set of the grids points $(x_{1j_1}, \cdots, x_{mj_m})$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$ is denoted by $Q^m_\Delta=\{(x_{1j_1}, \cdots, x_{mj_m})\}$. We use the notation $u_\Delta=\{u_{j_1,\cdots,j_m}\}$ to denote the discrete function with m indices defined on the grid domain Q^m_Δ.

Usually we have the abbreviations $\overset{+}{\Delta}_i u_{j_1,\cdots,j_i,\cdots,j_m}=(u_{j_1,\cdots,j_i+1,\cdots,j_m}-u_{j_1,\cdots,j_i,\cdots,j_m})/h_i$ and $\overline{\Delta}_i u_{j_1,\cdots,j_i,\cdots,j_m}=(u_{j_1,\cdots,j_i,\cdots,j_m}-u_{j_1,\cdots,j_i-1,\cdots,j_m})/h_i$. And then furthermore we adopt the notations $\delta^\alpha u_\Delta=\left\{\frac{\overset{+}{\Delta}{}^\alpha u_{j_1,\cdots,j_m}}{h^\alpha} \middle| j_i=0, 1, \cdots, J_i-\alpha_i;\ i=1, 2, \cdots, m\right\}$, where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ is a m-index and then $\delta^\alpha=\delta_1^{\alpha_1}\cdots\delta_m^{\alpha_m}$, $\overset{+}{\Delta}{}^\alpha=\overset{+}{\Delta}{}_1^{\alpha_1}\cdots\overset{+}{\Delta}{}_m^{\alpha_m}$, $h^\alpha=h_1^{\alpha_1}\cdots h_m^{\alpha_m}$, $|\alpha|=\alpha_1+\cdots+\alpha_m\geqslant 0$ and α_i $(i=1, 2, \cdots, m)$ are integers. Similarly we have

$$\delta_s^\beta u_{j_1,\cdots,j_s,\cdot}=\left\{\frac{\overset{+}{\Delta}{}^\beta u_{j_1,\cdots,j_s,j_{s+1},\cdots,j_m}}{h^\beta} \middle| j_i=0, 1, \cdots, J_i-\beta_i;\ i=s+1, \cdots, m\right\},$$

where $\beta=(\beta_{s+1}, \cdots, \beta_m)$ and $s=0, 1, \cdots$, m.

Now we adopt the following notations of the norms for the discrete functions and their difference quotients:

$$\begin{aligned} \|\delta^\alpha u_\Delta\|_{L_p(Q^m_\Delta)} &= \left(\sum_{i=1}^{m}\sum_{j_i=0}^{J_i-\alpha_i}\left|\frac{\overset{+}{\Delta}{}^\alpha u_{j_1,\cdots,j_m}}{h^\alpha}\right|^p h_1h_2\cdots h_m\right)^{\frac{1}{p}}, \quad 1\leqslant p<\infty; \\ \|\delta^\alpha u_\Delta\|_{L_\infty(Q^m_\Delta)} &= \max_{\substack{j_i=0,1,\cdots,J_i-\alpha_i \\ i=1,2,\cdots,m}}\left|\frac{\overset{+}{\Delta}{}^\alpha u_{j_1,\cdots,j_m}}{h^\alpha}\right| \end{aligned} \tag{3}$$

and

$$\begin{aligned} \|\delta_s^\beta u_{j_1,\cdots,j_s,\cdot}\|_{L_p(Q^{m-s}_\Delta)} &= \left(\sum_{i=s+1}^{m}\sum_{j_i=0}^{J_i-\beta_i}\left|\frac{\overset{+}{\Delta}{}^\beta u_{j_1,\cdots,j_m}}{h^\beta}\right|^p h_{s+1}\cdots h_m\right)^{\frac{1}{p}}, \quad 1\leqslant p<\infty; \\ \|\delta_s^\beta u_{j_1,\cdots,j_s,\cdot}\|_{L_\infty(Q^{m-s}_\Delta)} &= \max_{\substack{j_i=0,1,\cdots,J_i-\beta_i \\ i=s+1,\cdots,m}}\left|\frac{\overset{+}{\Delta}{}^\beta u_{j_1,\cdots,j_m}}{h^\beta}\right|, \quad s=0, 1, \cdots, m, \end{aligned} \tag{4}$$

where $\alpha=(\alpha_1, \cdots, \alpha_m)$, $\beta=(\beta_{s+1}, \cdots, \beta_m)$ $(s=0, 1, \cdots, m)$ and $|\alpha|\geqslant 0$, $|\beta|\geqslant 0$. Similarly we can define the seminorms and the norms for discrete functions in Sobolev spaces as follows: