Journal of Computational Mathematics
Vol.30, No.1, 2012, 47–58.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1110-m11si11

SOME RESIDUAL BOUNDS FOR APPROXIMATE EIGENVALUES AND APPROXIMATE EIGENSPACES*

Wen Li and Xiaoshan Chen
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
Email: liwen@scnu.edu.cn, xschen@scnu.edu.cn

Abstract

In this paper we consider approximate eigenvalues and approximate eigenspaces for the generalized Rayleigh quotient, and present some residual bounds. Our obtained bounds will improve the existing ones.

Mathematics subject classification: 65F10, 65F15, 65F35.
Key words: Approximate eigenvalue, Approximate eigenspace, Generalized Rayleigh quotient.

1. Introduction

By $\mathcal{C}^{m\times n}$ we denote the set of $m\times n$ complex matrices, by A^* we denote the conjugate transpose, and by I we denote the identity matrix. The Frobenius norm and the spectral norm of a matrix $\cdot$ are denoted by $\|\cdot\|_F$ and $\|\cdot\|_2$, respectively.

Let A and H be diagonalizable matrices with the following decompositions:

$$A = X\Lambda X^{-1} \equiv \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix} \text{and } H = Z\widetilde{\Lambda}Z^{-1}, \tag{1.1}$$

respectively, where $X\in\mathcal{C}^{n\times n}$, $Z\in\mathcal{C}^{m\times m}$, $X_1\in\mathcal{C}^{n\times m}$ $(m\le n)$,

$$\Lambda_1 = \mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_m), \quad \Lambda_2 = \mathrm{diag}(\lambda_{m+1},\lambda_{m+2},\cdots,\lambda_n),$$
$$\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\cdots,\widetilde{\lambda}_m).$$

Let A and H have the decomposition (1.1). Then δ_i is denoted by

$$\delta_i = \min_{\lambda\in\lambda(\Lambda_i),\widetilde{\lambda}\in\lambda(\widetilde{\Lambda})} |\lambda-\widetilde{\lambda}|, \quad i=1,2. \tag{1.2}$$

Notice that the decomposition (1.1) implies that

$$X^{-1} = \begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix}. \tag{1.3}$$

Let

$$R = AQ_1 - Q_1H \tag{1.4}$$

be the residual matrix of A with Q_1, where $A\in\mathcal{C}^{n\times n}$, $H\in\mathcal{C}^{m\times m}$ and $Q_1\in\mathcal{C}^{n\times m}$ $(m\le n)$, $\mathrm{rank}(Q_1)=m$. The spectrum of H is denoted by $\sigma(H)=\{\widetilde{\lambda}_1,\widetilde{\lambda}_2,\cdots,\widetilde{\lambda}_m\}$.

The quantity $||R||$ can be used to measure the difference between the spectrum $\sigma(H)$ and the spectrum $\sigma(\Lambda_1)$, and between the subspace $\Re(Q_1)$ and the approximate subspace $\Re(X_1)$. Some classical results in this topic are listed below:

* Received February 9, 2011 / Revised version received May 29, 2011 / Accepted May 31, 2011 / Published online January 9, 2012 /

1.1. Approximate eigenvalues

If A and H are Hermitian matrices and Q_1 has orthonormal columns, Kahan proved that there exists a permutation τ of $\langle m\rangle$ such that the following bound

$$\sum_{i=1}^{m}|\lambda_{\tau(i)}-\widetilde{\lambda}_i|^2\le 2||R||_F^2 \tag{1.5}$$

holds (e.g., see [17]), where $\langle m\rangle=\{1,2,...,m\}$.

If A is Hermitian and Q_1 has the orthonormal columns, $H=Q_1^*AQ_1$ is the Rayleigh quotient matrix, then it holds that [15]

$$\sum_{i=1}^{m}|\lambda_i-\widetilde{\lambda}_i|^2\le \frac{\|\sin\Theta(Q_1,X_1)\|_2^2}{1-\|\sin\Theta(Q_1,X_1)\|_2^2}||R||_F^2, \tag{1.6}$$

where the angle matrix $\Theta(Y,\widetilde{Y})$ between subspaces $\Re(Y)$ and $\Re(\widetilde{Y})$ is defined by

$$\Theta(Y,\widetilde{Y})=\arccos((Y^*Y)^{-\frac{1}{2}}Y^*\ \widetilde{Y}\ (\widetilde{Y}^*\widetilde{Y})^{-1}\ \widetilde{Y}^*\ Y(Y^*Y)^{-\frac{1}{2}})^{\frac{1}{2}}\,,$$

Y and $\widetilde{Y}\in\mathcal{C}^{n\times k}(n>k)$ are full column rank matrices. In particular, if Y and $\widetilde{Y}\in\mathcal{C}^{n\times k}(n>k)$ have orthonormal columns, then for any unitarily invariant norm $||\cdot||$ we have

$$||\sin\Theta(Y,\widetilde{Y})||=||(\widetilde{Y}_c)^*Y||, \tag{1.7}$$

where $(\widetilde{Y},\widetilde{Y}_c)$ is an $n\times n$ unitary matrix (e.g., see [13]).

If A and H are diagonalizable matrices with the decomposition (1.1), and Q_1 has full column rank, then Liu [11] obtained a result as follows: There exists a permutation τ of $\langle m\rangle$ such that

$$\sigma_{\min}^2(Q_1)\sum_{i=1}^{m}|\lambda_{\tau(i)}-\widetilde{\lambda}_i|^2\le\kappa^2(X)\kappa^2(Z)||R||_F^2\,, \tag{1.8}$$

where $\sigma_{\min}(Q_1)$ denotes the smallest singular value of Q_1. In particular, if A and H are Hermitian matrices, then

$$\sigma_{\min}^2(Q_1)\sum_{i=1}^{m}|\lambda_{\tau(i)}-\widetilde{\lambda}_i|^2\le||R||_F^2\,. \tag{1.9}$$

It is easy to see that the bound (1.9) generalizes the one in (1.5).

1.2. Approximate eigenspaces

If A and H are Hermitian matrices and Q_1 has orthonormal columns, Kahan and Davis [1] obtained a well-known result, i.e., $\sin\Theta$ Theorem:

$$\|\sin\Theta(Q_1,X_1)\|_F\le\frac{\|R\|_F}{\delta_2} \tag{1.10}$$

provided $\delta_2>0$, where δ_2 is given by (1.2). If A and H are Hermitian matrices, and Q_1 is a full column rank matrix, then (see, e.g., [13])

$$\sigma_{\min}(Q_1)\|\sin\Theta(Q_1,X_1)\|_F\le\frac{\|R\|_F}{\delta_2} \tag{1.11}$$