Journal of Computational Mathematics
Vol.30, No.2, 2012, 157–176.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1109-m3491

A PROJECTION METHOD WITH REGULARIZATION FOR THE CAUCHY PROBLEM OF THE HELMHOLTZ EQUATION*

Yunyun Ma, Fuming Ma and Heping Dong
School of Mathematics, Jilin University, Changchun 130012, China
Email: mayy2009@gmail.com, mafm@jlu.edu.cn, dhp@jlu.edu.cn

Abstract

This paper is concerned with the reconstruction of the radiation wave field in the exterior of a bounded two- or three-dimensional domain from the knowledge of Cauchy data on a part of the boundary of the aforementioned domain. It is described by the Cauchy problem for the Helmholtz equation. By using the Dirichlet-to-Neumann map, this problem is transformed into an operator equation with compact operator. We rigorously justify the asymptotic behaviors of singular values of the compact operator. Then a projection method with regularization is applied to solve the operator equation, and the convergence of the regularization method is discussed. Finally, several numerical examples are presented to illustrate the approach. The results demonstrate that the algorithm is effective.

Mathematics subject classification: 35R25, 35R30, 78A40.
Key words: Helmholtz equation, Cauchy Problem, Projection method, Regularization.

1. Introduction

The Helmholtz equation arises naturally in many physical applications related to wave propagation and vibration phenomena (see, e.g., [3, 6, 7, 19] and the references therein). It is often used to describe the vibration of a structure, the radiation wave and the scattering of a wave. We focus on the determination of the radiation wave field in this study.

The theoretical and numerical studies on the Helmholtz equation have been developed extensively in the past century. Most of the results studying on the numerical methods of the Helmholtz equation are related with the boundary value problems, i.e., the Dirichlet, Neumann or mixed boundary value problems (see, e.g., [1, 4, 10, 12, 14]). The well-posedness of the boundary value problems of the Helmholtz equation via the removal of the eigenvalues of the Laplacian operator is well established. Unfortunately, many engineering problems do not belong to this category. In particular, the boundary conditions are often incomplete, either in the form of the underspecified and overspecified boundary conditions on the different parts of the boundary or the solution is prescribed at some internal points in the domain. These problems are usually ill-posed, i.e., the existence, uniqueness and stability of their solutions are not always guaranteed.

Motivated by the advance in the measurement technology, the wave field and its gradient can be collected at a portion of a closed surface. Therefore, many important studies have been devoted to the Cauchy problem associated with the Helmholtz equation, which is severely ill-posed [13]. The determination of the sources was discussed in [9]. The reconstruction of the radiation field was discussed in [26]. Unlike the boundary value problems, the uniqueness of

* Received August 16, 2010 / Revised version received July 11, 2011 / Accepted July 29, 2011 /
Published online February 24, 2012 /

the Cauchy problem is guaranteed without the necessity of removing the eigenvalues for the Laplacian. However, the Cauchy problem suffers from the non-existence and instability of the solution. A number of numerical methods have been proposed to solve this problem, such as the iterative algorithm proposed by Kozlov et al. [17,18,21,22], the spherical wave expansion method [30,31], the Fourier regularization method [8,27], the method of fundamental solution [23], the boundary knot method [15], the plane wave method [16], the boundary element-minimal error method [24], and the moment method [28,29].

In this paper, we propose a numerical method dealing with the Cauchy problem for the Helmholtz equation in the exterior of a bounded two- or three-dimensional domain. The paper is organized as follows. In Section 2, we formulate the problem and transform the Cauchy problem into an operator equation with a compact operator. In Section 3, we discuss the regularity properties of the compact operator and propose a projection method with regularization for solving the compact operator equation in the two-dimensional case. Section 4 provides the projection method with regularization for the reconstruction of the radiation wave field in the three-dimensional case, which is a little different from the two-dimensional case. In Section 5, we show several numerical examples to demonstrate the effectiveness of our method. Finally, a short conclusion in Section 6 summarizes the results of this paper.

2. Mathematical Formulation of the Problem

Consider the reconstruction of the radiation wave field arising from the sources of radiation. Let B_R be a ball of radius R centered at the origin in $\mathbb{R}^d$ $(d = 2, 3)$. It contains all the sources of the wave field. When a time harmonic wave is considered, the propagation of the waves in the homogeneous isotropic medium is governed by the Helmholtz equation

$$\triangle u + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_R}, \tag{2.1}$$

with the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \Big(\frac{\partial u}{\partial r} - \mathrm{i}ku \Big) = 0, \quad r = |x|, \tag{2.2}$$

where $k = w/c > 0$ is the wave number with the angular frequency w and the speed of sound c. $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

In this paper, both the wave field f and its normal derivative g on Γ, where $\Gamma \subset \partial B_R$ is an open set, are considered as the input data for the reconstruction of the radiation wave field in the domain $\mathbb{R}^d \setminus B_R$. Hence, the problem can be formulated as finding a radiation solution $u \in C^2(\mathbb{R}^d \setminus \overline{B_R}) \cap C(\mathbb{R}^d \setminus B_R)$ of the equation (2.1)–(2.2), which satisfies the Cauchy condition

$$u = f, \quad \frac{\partial u}{\partial n} = g, \quad \text{on } \Gamma, \tag{2.3}$$

where $\vec{n}$ is the unit outward normal of the boundary Γ.

Suppose the Cauchy problem (2.1)–(2.3) has a solution $u \in C^2(\mathbb{R}^d \setminus \overline{B_R}) \cap C(\mathbb{R}^d \setminus B_R)$. Denote $u = \tilde{f}$ on ∂B_R. It is well known that if $\tilde{f}$ on ∂B_R is given, the radiation solution u of the Helmholtz equation (2.1) satisfying the radiation condition (2.2) is unique determined (see, e.g., [2, Chapter 3]). Let Λ be the Dirichlet-to-Neumann map, which is defined by

$$\Lambda \tilde{f} = \frac{\partial u}{\partial n}\Big|_{\partial B_R}.$$