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HOMOTOPY CURVE TRACKING FOR TOTAL VARIATION IMAGE RESTORATION*

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Abstract

The total variation (TV) minimization problem is widely studied in image restoration. Although many alternative methods have been proposed for its solution, the Newton method remains not usable for the primal formulation due to no convergence. A previous study by Chan, Zhou and Chan [15] considered a regularization parameter continuation idea to increase the domain of convergence of the Newton method with some success but no robust parameter selection schemes. In this paper, we consider a homotopy method for the same primal TV formulation and propose to use curve tracking to select the regularization parameter adaptively. It turns out that this idea helps to improve substantially the previous work in efficiently solving the TV Euler-Lagrange equation. The same idea is also considered for the two other methods as well as the deblurring problem, again with improvements obtained. Numerical experiments show that our new methods are robust and fast for image restoration, even for images with large noisy-to-signal ratio.

Mathematics subject classification: 65N06, 65B99.
Key words: Image restoration, Total variation, Newton method, Homotopy method, Correction and curve tracking.

1. Introduction

It is well known that an image can often become blurry and noisy if corrupted during formation, transmission or recording process. This degradation makes it difficult to do further image processing tasks such as edge detection, pattern recognition, and object tracking, etc. Denote by z the observed image (known) and u the desired true image (unknown), both defined on a bounded convex region Ω of $\mathbb{R}^d$ (for simplicity we will assume Ω to be a square in $\mathbb{R}^2$). Consider the common degradation model

$$z = Ku + \eta, \tag{1.1}$$

where η is an additive noise term (also unknown) and K is a known linear operator representing the blur (usually a convolution), the image is only corrupted by noise when K is the identity. We wish to reconstruct the true image u from the observed image z.

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There are many different modeling methods proposed to obtain an estimate of u [28]. One effective and well-known method is the total variation-based method by Rudin, Osher and Fatemi [26], consisting of solving the following constrained optimization problem:

$$\min_u \int_\Omega |\nabla u| dxdy \quad \text{subject to} \quad \|Ku - z\|^2 = \sigma^2. \tag{1.2}$$

Here $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$, $\|\cdot\|$ is the norm in $\mathbf{L}^2(\Omega)$ and σ is the standard deviation of the noise η. This problem is naturally linked to the following unconstrained problem – the minimization of the total variation penalized least squares functional (see [8, 26, 28]):

$$\alpha \int_\Omega |\nabla u| dxdy + \frac{1}{2}\|Ku - z\|^2, \tag{1.3}$$

where α is a positive parameter controlling the trade-off between goodness of fit-to-the-data and variability in u. The main advantage of the total variation restoration models is that their solutions preserve edges very well. But other models without the TV are also effective [5,13,14].

In spite of the fact that the variational problem (1.3) is convex, the computation is not easy since the total variation semi-norm is a nonlinear nondifferentiable functional. To overcome the nondifferentiation difficulty, one approach is the dual methods (see [3, 7]) and the other is a split Bregman iteration [33]. However, the commonly used technique is to approximate the term $|\nabla u|$ by $\sqrt{|\nabla u|^2 + \beta}$, where β is a small positive parameter, and the unconstrained minimization problem (1.3) becomes

$$\min_u \left\{ f(u) = \alpha \int_\Omega \sqrt{|\nabla u|^2 + \beta} dxdy + \frac{1}{2}\|Ku - z\|^2 \right\}. \tag{1.4}$$

It is shown in [1] that the solution of (1.4) converges to the solution of (1.3) when $\beta \to 0$. The corresponding Euler-Lagrange partial differential equation (PDE) for (1.4) is

$$g(u) = -\alpha \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}} \right) + K^*(Ku - z) = 0, \quad (x, y) \in \Omega, \tag{1.5}$$

with homogeneous Neumann boundary condition $\partial u / \partial \vec{n} = 0$, $(x, y) \in \partial\Omega$. Here $\nabla \cdot$ is the divergence operator, K^* is the adjoint operator of K with respect to the $\mathbb{L}^2$ inner product, $\partial\Omega$ is the boundary of Ω and $\vec{n}$ is the normal vector of $\partial\Omega$. It should be remarked that even for moderately small β, the Newton method does not converge with the common starting iterate $u = z$; therefore, one cannot find any use of Newton type methods for this primal equation in the literature.

Before we present a method to help the Newton method, we briefly review four categories of methods for solving (1.5).

1) **Gradient descent methods** [24, 26]. As used in Rudin *et al.* [26], instead of the elliptic PDE, a parabolic PDE with time as an evolution parameter is solved by the gradient descent method

$$u_t = \mathcal{N}(u) \equiv \alpha \nabla \cdot \Big(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}} \Big) - K^*(Ku - z), \quad u(x, y, 0) = z. \tag{1.6}$$

An accelerated version is $u_t = |\nabla u| \mathcal{N}(u)$ as first used by [24]. This method is preferred in many situations for its simplicity, user-independent choice of regularize parameter and fast