Journal of Computational Mathematics
Vol.30, No.4, 2012, 354–380.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1111-m3776

REGULARIZATION METHODS FOR THE NUMERICAL SOLUTION OF THE DIVERGENCE EQUATION $\nabla \cdot \mathbf{u} = f$*

Alexandre Caboussat
University of Houston, Department of Mathematics, Houston, Texas, USA
Email: caboussat@math.uh.edu
Haute Ecole de Gestion, Genève, Switzerland
Email: alexandre.caboussat@hesge.ch
Roland Glowinski
University of Houston, Department of Mathematics, Houston, Texas, USA
Email: roland@math.uh.edu

Abstract

The problem of finding a L^∞-bounded two-dimensional vector field whose divergence is given in L^2 is discussed from the numerical viewpoint. A systematic way to find such a vector field is to introduce a non-smooth variational problem involving a L^∞-norm. To solve this problem from calculus of variations, we use a method relying on a well-chosen augmented Lagrangian functional and on a mixed finite element approximation. An Uzawa algorithm allows to decouple the differential operators from the nonlinearities introduced by the L^∞-norm, and leads to the solution of a sequence of Stokes-like systems and of an infinite family of local nonlinear problems. A simpler method, based on a L^2-regularization is also considered. Numerical experiments are performed, making use of appropriate numerical integration techniques when non-smooth data are considered; they allow to compare the merits of the two approaches discussed in this article and to show the ability of the related methods at capturing L^∞-bounded solutions.

Mathematics subject classification: 65N30, 65K10, 65J20, 49K20, 90C47.
Key words: Divergence equation, Bounded solutions, Regularization methods, Augmented Lagrangian, Uzawa algorithm, Nonlinear variational problems.

1. Introduction and Motivations

The purpose of this article is to investigate the numerical solution of the following problem

$$\text{Find } \mathbf{u} \in (L^\infty(\Omega) \cap W^{1,p}(\Omega))^2 \text{ such that } \nabla \cdot \mathbf{u} = f \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.1}$$

where $f \in L^p(\Omega)$ is given. This problem is under-determined in the sense that the solution is defined up to the addition of an arbitrary function with zero curl. It is common to look for a solution that is the *gradient of a potential function* (as in electromagnetism for example). The resulting potential function is therefore the solution of a Poisson equation.

However, when $p = 1$ or $p = +\infty$, obtaining a solution which is the gradient of a potential function is not necessarily possible, see, e.g., [1, 2]. Moreover, when considering $p = 2$, the gradient of such a potential function obtained by solving a Poisson equation is not necessarily bounded [3]. Therefore, we focus hereafter on the so-called *non-smooth case* that consists in enforcing bounded solutions instead of gradients of potential functions.

* Received June 8, 2011 / Revised version received November 15, 2011 / Accepted December 14, 2011 / Published online July 6, 2012 /

This problem has been studied from the theoretical viewpoint in [1, 2, 4], with a particular emphasis on the torus domain, using arguments from [5]. Regularity issues have been discussed in [6, 7]. The case $p = 1$ is partially discussed in [8]. In [1], it is shown that one can actually replace $L^\infty(\Omega)$ by $C^0\left(\overline{\Omega}\right)$ in (1.1) if $p = 2$.

In order to search for a bounded solution, we introduce an equivalent variational formulation. More precisely, for $g > 0$ a given parameter and $f \in L^p(\Omega)$ given, we look for the solution of

$$\inf_{\mathbf{v}\in\mathbf{E}_f} \left[\frac{1}{p} \int_\Omega |\nabla \mathbf{v}|^p \, d\mathbf{x} + g \, ||\mathbf{v}||_\infty \right], \tag{1.2}$$

where $||\mathbf{v}||_\infty := \operatorname{ess\,sup}_{\mathbf{x}\in\Omega} \sqrt{v_1^2 + v_2^2}$, for all $\mathbf{v} = \{v_1, v_2\}$ and

$$\mathbf{E}_f = \left\{ \mathbf{v} \in (W^{1,p}(\Omega) \cap L^\infty(\Omega))^2 \ , \ \nabla \cdot \mathbf{v} = f \text{ in } \Omega \right\}.$$

This choice of the objective function allows to enforce the appropriate regularity of the solution. The minimizer of this constrained variational problem provides a solution to the divergence equation (1.1) with the appropriate regularity, and allows to "fix the constant" in the family of solutions of the divergence equation. From now on, we focus on the case $p = 2$ ($f \in L^2(\Omega)$). Actually for some test problems, we will assume that $f \in L^p(\Omega)$ with $1 \le p < 2$.

Numerical methods for such non-smooth variational problems require an appropriate treatment of the non-Hilbertian features introduced by the sup-norm. Such numerical algorithms for non-smooth problems have been developed in the framework of fully nonlinear elliptic problems [9, 10], or for generalized eigenvalue problems [11–14].

We advocate an augmented Lagrangian algorithm that allows to decouple the solution of a non-smooth variational problem into the solution of a sequence of Stokes-like systems (solved for instance with stabilized continuous finite elements [15, 16]), and non-smooth problems solved locally (namely at each grid point of a finite element triangulation). The treatment of the sup-norm is achieved with a *duality* approach that has already been successfully applied in [17].

In a second part, we will address a L^2-regularization of problem (1.1) and compare with the previous approach. Namely, for $\gamma > 0$, we look for a solution of

$$\inf_{\mathbf{v}\in\mathbf{T}_f} \left[\frac{1}{2} \int_\Omega |\nabla \mathbf{v}|^2 \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega |\mathbf{v}|^2 \, d\mathbf{x} \right] \tag{1.3}$$

with

$$\mathbf{T}_f = \left\{ \mathbf{v} \in (H^1(\Omega))^2 \ , \ \nabla \cdot \mathbf{v} = f \text{ in } \Omega \right\}.$$

This variational problem leads to the solution of a Stokes system.

Regularization methods are quite common in the literature as basic tools for the solution of ill-posed problems. They are well-known in the framework of inverse problems, starting with [18–21]. In [22, 23], classical questions such as the appropriate choice of parameters and generalizations to family of regularization methods have been addressed. Many advances have been recently made when relying on non-smooth regularization terms using L^1 or L^∞ norms (or their algebraic equivalents), see, e.g., [24, 25] This approach has already been used by the authors in the framework of *non-smooth* problems, see, e.g., [17, 26].

This article is organized as follows: Section 2 details the generic model problem and provides some existence results as well as the description of some properties of the solution of (1.2). In Section 3, an augmented Lagrangian algorithm *à la Uzawa* is described. The discrete equivalent of this algorithm, obtained after discretization with continuous mixed piecewise linear finite elements, is detailed in Section 4. Numerical experiments with the L^∞-regularization are performed in Section 5, for smooth and non-smooth data, and a computational investigation of the convergence of the approximations (with respect to the mesh size) is achieved. Section 6 details the L^2-regularization method, and presents numerical results to compare both approaches.