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OPTIMAL CONTROL OF THE LAPLACE–BELTRAMI OPERATOR ON COMPACT SURFACES: CONCEPT AND NUMERICAL TREATMENT*

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Abstract

We consider optimal control problems of elliptic PDEs on hypersurfaces Γ in $\mathbb{R}^n$ for $n = 2, 3$. The leading part of the PDE is given by the Laplace-Beltrami operator, which is discretized by finite elements on a polyhedral approximation of Γ. The discrete optimal control problem is formulated on the approximating surface and is solved numerically with a semi-smooth Newton algorithm. We derive optimal a priori error estimates for problems including control constraints and provide numerical examples confirming our analytical findings.

Mathematics subject classification: 58J32, 49J20, 49M15.
Key words: Elliptic optimal control problem, Laplace-Beltrami operator, Surfaces, Control constraints, Error estimates, Semi-smooth Newton method.

1. Introduction

We are interested in the numerical treatment of the following linear-quadratic optimal control problem on a n-dimensional, sufficiently smooth hypersurface $\Gamma \subset \mathbb{R}^{n+1}$, $n = 1, 2$.

$$
\begin{gathered}
\min_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} J(u, y) = \frac{1}{2}\|y - z\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \\
\text{subject to} \quad u \in U_{ad} \quad \text{and} \\
\int_\Gamma \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{c} y \varphi \, \mathrm{d}\Gamma = \int_\Gamma u \varphi \, \mathrm{d}\Gamma \,, \forall \varphi \in H^1(\Gamma)
\end{gathered}
\tag{1.1}
$$

with $U_{ad} = \{v \in L^2(\Gamma) \mid a \le v \le b\}$, $a < b \in \mathbb{R}$. For simplicity we will assume Γ to be compact and $\mathbf{c} = 1$. In section 4 we briefly investigate the case $\mathbf{c} = 0$, in section 5 we give an example on a surface with boundary.

Problem (1.1) may serve as a mathematical model for the optimal distribution of surfactants on a biomembrane Γ with regard to achieving a prescribed desired concentration z of a quantity y.

It follows by standard arguments that (1.1) admits a unique solution $u \in U_{ad}$ with unique associated state $y = y(u) \in H^2(\Gamma)$.

Our numerical approach uses variational discretization applied to (1.1), see [9] and [10], on a discrete surface Γ^h approximating Γ. The discretization of the state equation in (1.1) is achieved

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by the finite element method proposed in [4], where a priori error estimates for finite element approximations of the Poisson problem for the Laplace-Beltrami operator are provided. Let us mention that uniform estimates are presented in [2], and steps towards a posteriori error control for elliptic PDEs on surfaces are taken by Demlow and Dziuk in [3]. For alternative approaches for the discretization of the state equation by finite elements see the work of Burger [1]. Finite element methods on moving surfaces are developed by Dziuk and Elliott in [5]. To the best of the authors knowledge, the present paper contains the first attempt to treat optimal control problems on surfaces.

We assume that Γ is of class C^2. As an embedded, compact hypersurface in $\mathbb{R}^{n+1}$ it is orientable with an exterior unit normal field ν and hence the zero level set of a signed distance function d such that

$$|d(x)| = \operatorname{dist}(x, \Gamma) \quad \text{and} \quad \nu(x) = \frac{\nabla d(x)}{\|\nabla d(x)\|} \quad \text{for } x \in \Gamma.$$

Further, there exists an neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of Γ, such that d is also of class C^2 on $\mathcal{N}$ and the projection

$$a : \mathcal{N} \to \Gamma\,, \quad a(x) = x - d(x)\nabla d(x) \tag{1.2}$$

is unique, see e.g. [6, Lemma 14.16]. Note that $\nabla d(x) = \nu(a(x))$.

Using a we can extend any function $\phi : \Gamma \to \mathbb{R}$ to $\mathcal{N}$ as $\bar{\phi}(x) = \phi(a(x))$. This allows us to represent the surface gradient in global exterior coordinates $\nabla_\Gamma \phi = (I - \nu\nu^T)\nabla\bar{\phi}$, with the euclidean projection $(I - \nu\nu^T)$ onto the tangential space of Γ.

We use the Laplace-Beltrami operator $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ in its weak form i.e. $\Delta_\Gamma : H^1(\Gamma) \to H^1(\Gamma)^*$

$$y \mapsto -\int_\Gamma \nabla_\Gamma y \nabla_\Gamma(\,\cdot\,)\,\mathrm{d}\Gamma \in H^1(\Gamma)^*\,.$$

Let S denote the prolongated restricted solution operator of the state equation

$$S : L^2(\Gamma) \to L^2(\Gamma)\,, \quad u \mapsto y \qquad -\Delta_\Gamma y + \mathbf{c}y = u\,,$$

which is compact and constitutes a linear homeomorphism onto $H^2(\Gamma)$, see [4, 1. Theorem].

By standard arguments we get the following necessary (and here also sufficient) conditions for optimality of $u \in U_{ad}$

$$\begin{aligned} &\langle \nabla_u J(u, y(u)), v - u\rangle_{L^2(\Gamma)} \\ =&\langle \alpha u + S^*(Su - z), v - u\rangle_{L^2(\Gamma)} \ge 0 \quad \forall v \in U_{ad}\,. \end{aligned} \tag{1.3}$$

We rewrite (1.3) as

$$u = \mathrm{P}_{U_{ad}}\left(-\frac{1}{\alpha}S^*(Su - z)\right)\,, \tag{1.4}$$

where $\mathrm{P}_{U_{ad}}$ denotes the L^2-orthogonal projection onto U_{ad}.

2. Discretization

We now discretize (1.1) using an approximation Γ^h to Γ which is globally of class $C^{0,1}$. Following Dziuk, we consider polyhedral $\Gamma^h = \bigcup_{i\in I_h} T_h^i$ consisting of triangles T_h^i with corners on Γ, whose maximum diameter is denoted by h. With FEM error bounds in mind we assume