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GENERALIZED PRECONDITIONED HERMITIAN AND SKEW-HERMITIAN SPLITTING METHODS FOR NON-HERMITIAN POSITIVE-DEFINITE LINEAR SYSTEMS*

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Abstract

In this paper, a generalized preconditioned Hermitian and skew-Hermitian splitting (GPHSS) iteration method for a non-Hermitian positive-definite matrix is studied, which covers standard Hermitian and skew-Hermitian splitting (HSS) iteration and also many existing variants. Theoretical analysis gives an upper bound for the spectral radius of the iteration matrix. From practical point of view, we have analyzed and implemented inexact generalized preconditioned Hermitian and skew-Hermitian splitting (IGPHSS) iteration, which employs Krylov subspace methods as its inner processes. Numerical experiments from three-dimensional convection-diffusion equation show that the GPHSS and IGPHSS iterations are efficient and competitive with standard HSS iteration and AHSS iteration.

Mathematics subject classification: 65F10, 65F20, 65F30, 65F50.
Key words: Hermitian and skew-Hermitian splitting, Iteration method, Inner iteration.

1. Introduction

We consider the solution of large sparse system of linear equations

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \qquad x, b \in \mathbb{C}^n, \tag{1.1}$$

where A is a non-Hermitian and positive definite matrix. Bai, et al. [10] first presented the Hermitian and skew-Hermitian splitting (HSS) iteration method, which is based on the Hermitian and skew-Hermitian splitting

$$A = H + S, \tag{1.2}$$

where

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*) \tag{1.3}$$

is the Hermitian and the skew-Hermitian parts of A, respectively.

Let α be a positive constant. The standard HSS iterative scheme works as follows: Given an arbitrary initial guess $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.4}$$

where α is a given positive constant.

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When H is positive definite, the HSS iteration method is unconditionally convergent for all $\alpha > 0$ and for any choice of $x^{(0)}$. Moreover, an upper bound on the spectral radius of the iteration matrix is given, which can be minimized by choosing $\alpha = \sqrt{\lambda_1 \lambda_n}$, where λ_1 and λ_n are the minimal and the maximal eigenvalues of H, respectively. In order to save the computational cost, the authors considered inexact HSS method with Krylov solvers in the inner iterations, see also [10, 12]. Further, Bai, Golub and Ng in [10] carefully studied the asymptotic convergence rates and the optimal choices of the inner iteration steps for two specific kinds of inexact HSS iteration.

HSS iteration method immediately attracted considerable attention and resulted in many papers devoted to various aspects of the new algorithms. For instance, generalized Hermitian and skew-Hermitian splitting (GHSS) iteration in [14]; preconditioned Hermitian and skew-Hermitian splitting (PHSS) iteration in [8, 13, 14]; lopsided Hermitian and skew-Hermitian splitting (LHSS) iteration in [22]; asymmetric Hermitian and skew-Hermitian splitting (AHSS) iteration in [21]; positive-definite and skew-Hermitian splitting (PSS) iteration and block triangular and skew-Hermitian splitting (BTSS) iterations in [9].

On the other hand, HSS iteration method was successfully extended to the solution of saddle point problem in [16] and preconditioning techniques for Krylov subspace methods [17, 19]; see also [6, 20, 24]. It was noted in [17] that the method can unconditionally convergent when the Hermitian part H is positive semidefinite for the special case of (generalized) saddle point problems. Other developments including studies on the optimal selection of iteration parameters, successive overrelaxation acceleration, extension to certain singular systems and applications of the HSS preconditioner have been well established in [4, 5, 7, 8, 11, 15, 18] and the references therein.

In this paper, based on the splitting (1.2)-(1.3), we generalize the HSS iteration scheme into a new approach, called generalized preconditioned HSS (GPHSS) iteration. By introducing two symmetric positive definite matrices P_1 and P_2, the GPHSS iterative scheme works as follows.

Method 1.1. The GPHSS iteration method. *Given an arbitrary initial guess $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until $x^{(k)}$ converges, compute*

$$\begin{cases} (\alpha P_1 + H)x^{(k+\frac{1}{2})} = (\alpha P_1 - S)x^{(k)} + b, \\ (\beta P_2 + S)x^{(k+1)} = (\beta P_2 - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.5}$$

where α and β are given positive constants.

Note that the GPHSS iteration method can cover many existing variants of the standard HSS iteration. For instance, when $\alpha = \beta$ and $P_1 = P_2 = I$, the GPHSS iteration method is equivalent to the standard HSS iteration method in [10]; when $\alpha = 0$ and $P_2 = I$, it leads to the LHSS iteration method in [22]; when $\alpha \neq \beta$ and $P_1 = P_2 = I$, it results in AHSS iteration method in [21] and when $P_1 = P_2$, it is PHSS iteration method in [8, 13].

Theoretical analysis gives an upper bound about the contraction factor of GPHSS iteration method, which shows the relations among GPHSS, HSS and other existing variants. From practical point of view, we also analyze the convergence theory of inexact variants of the GPHSS iteration method and their implementation. A number of numerical experiments from discrete three-dimensional convection-diffusion equation are presented to illustrate the advantages of the GPHSS iteration method.

This paper is organized as follow. We study the convergence properties of the GPHSS iteration method in Section 2. In Section 3, we discuss in detail the implementation of GPHSS