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THE BEST L^2 NORM ERROR ESTIMATE OF LOWER ORDER FINITE ELEMENT METHODS FOR THE FOURTH ORDER PROBLEM*

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Abstract

In the paper, we analyze the L^2 norm error estimate of lower order finite element methods for the fourth order problem. We prove that the best error estimate in the L^2 norm of the finite element solution is of second order, which can not be improved generally. The main ingredients are the saturation condition established for these elements and an identity for the error in the energy norm of the finite element solution. The result holds for most of the popular lower order finite element methods in the literature including: the Powell-Sabin C^1-P_2 macro element, the nonconforming Morley element, the C^1-Q_2 macro element, the nonconforming rectangle Morley element, and the nonconforming incomplete biquadratic element. In addition, the result actually applies to the nonconforming Adini element, the nonconforming Fraeijs de Veubeke elements, and the nonconforming Wang-Xu element and the Wang-Shi-Xu element provided that the saturation condition holds for them. This result solves one long standing problem in the literature: can the L^2 norm error estimate of lower order finite element methods of the fourth order problem be two order higher than the error estimate in the energy norm?

Mathematics subject classification: 65N10, 65N15, 35J25.
Key words: L^2 norm error estimate, Energy norm error estimate, Conforming, Nonconforming, The Kirchhoff plate.

1. Introduction

We shall consider the L^2 norm error estimate of the finite element method of the Kirchhoff plate bending problem reads: Given $g \in L^2(\Omega)$ find $w \in W := H_0^2(\Omega)$ with

$$a(w,v) = (g,v)_{L^2(\Omega)} \quad \text{for all } v \in W\,. \tag{1.1}$$

The bilinear form $a(w,v)$ reads

$$a(w,v) : = (\nabla^2 w, \nabla^2 v)_{L^2(\Omega)} \text{ for any } w\,, v \in W\,, \tag{1.2}$$

where $\nabla^2 w$ is the Hessian of w. For this fourth order elliptic problem, there are a number of conforming/nonconforming finite element methods in the literature, see for instance, [6, 8, 20] and the references therein.

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Let W_h be some conforming or nonconforming finite element space defined over the triangulation $\mathcal{T}_h$ of the domain $\Omega \subset \mathbb{R}^2$ into rectangles or triangles, the discrete problem reads: Find $w_h \in W_h$ such that

$$a_h(w_h, v_h) = (g, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in W_h\,. \tag{1.3}$$

The broken version $a_h(\cdot,\cdot)$ follows

$$a_h(w_h, v_h) := (\nabla_h^2 w_h, \nabla_h^2 v_h)_{L^2(\Omega)} \text{ for any } w_h\,, v_h \in W + W_h\,,$$

where ∇_h^2 is the discrete counterpart of the Hessian operator ∇^2, which is defined elementwise with respect to the triangulation $\mathcal{T}_h$ since W_h may be nonconforming. If $W_h \subset W$, we have $a_h(w_h, v_h) = a(w_h, v_h)$ for $w_h, v_h \in W_h$.

Under some continuity condition of the discrete space W_h [6, 8, 20], the discrete problem (1.3) will be well-posed and consequently admit a unique solution. Define the residual

$$\mathrm{Res}_h(v_h) := (g, v_h)_{L^2(\Omega)} - a_h(w, v_h) \text{ for any } v_h \in W_h\,. \tag{1.4}$$

Then we have the following Strang Lemma:

$$\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)} \le C\Big(\sup_{v_h\in W_h} \frac{\mathrm{Res}_h(v_h)}{\|\nabla_h^2 v_h\|_{L^2(\Omega)}} + \min_{v_h\in W_h} \|\nabla_h^2(w - v_h)\|_{L^2(\Omega)}\Big). \tag{1.5}$$

Here and throughout this paper C is some generic positive constant which is independent of the meshsize. We are interested in some lower order methods: the nonconforming Morley element [15, 17, 21], the Powell-Sabin C^1– P_2 macro element [18], the C^1– Q_2 macro element [10], the nonconforming rectangle Morley element [24], and the nonconforming incomplete biquadratic element [16, 27]. For these discrete methods, it follows from the Strang Lemma that

$$\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)} \le Ch\|g\|_{L^2(\Omega)}, \tag{1.6}$$

provided that $w \in H^3(\Omega) \cap H_0^2(\Omega)$. Here and throughout this paper, h denotes the meshsize which is defined by

$$h := \max_{K\in\mathcal{T}_h} h_K \text{ with } h_K \text{ the diameter of } K. \tag{1.7}$$

By the dual argument, we have

$$\|w - w_h\|_{L^2(\Omega)} + \|\nabla_h(w - w_h)\|_{L^2(\Omega)} \le Ch^2\|g\|_{L^2(\Omega)}, \tag{1.8}$$

provided that Ω is smooth or convex, where ∇_h is the elementwise defined counterpart of the gradient operator ∇. By the approximation property of the discrete space, we have

$$\inf_{v_h\in W_h} \|w - v_h\|_{L^2(\Omega)} \le Ch^3|w|_{H^3(\Omega)}, \tag{1.9}$$

for all the methods under consideration. Compared to the approximation result (1.9), the L^2 norm error estimate in (1.8) is obviously not optimal. Then one long standing problem for the finite element method of the fourth order problem is: can the L^2 norm error estimate of lower order finite element methods of the fourth order problem be two order higher than the error estimate in the energy norm? The aim of the paper is to prove that the L^2 norm error estimate in (1.8) can not be improved for these methods under consideration. The main ingredients are the saturation condition and the identity of the error in the energy norm.