Journal of Computational Mathematics
Vol.30, No.5, 2012, 461–482.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1201-m3848

PRECONDITIONING THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VARIABLE VISCOSITY*

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Abstract

This paper deals with preconditioners for the iterative solution of the discrete Oseen problem with variable viscosity. The motivation of this work originates from numerical simulations of multiphase flow, governed by the coupled Cahn-Hilliard and incompressible Navier-Stokes equations. The impact of variable viscosity on some known preconditioning technique is analyzed. Theoretical considerations and numerical experiments show that some broadly used preconditioning techniques for the Oseen problem with constant viscosity are also efficient when the viscosity is varying.

Mathematics subject classification: 65F10, 65F08, 65N30.
Key words: Navier-Stokes equations, Saddle point systems, Augmented Lagrangian, Finite elements, Iterative methods, Preconditioning.

1. Introduction

In this paper we consider preconditioned iterative solution methods for the stationary incompressible Navier-Stokes (N-S) equations with variable viscosity. Here we assume that the kinematic viscosity coefficient is a smooth function, such that

$$0 < \nu_{\min} \leq \nu(\mathbf{x}) \leq \nu_{\max},$$

where $\nu_{\min}$ and $\nu_{\max}$ denote its minimal and maximal value. Many mathematical models in fluid dynamics involve non-constant viscosity. For example, viscosity is a function of the temperature in convection flows (e.g., [19, 44]); it is a function of pressure and the rate-of-strain tensor in non-Newtonian flows (e.g., [11, 42]). In some quasi-Newtonian flows the variable viscosity may also depend on pressure and shear (e.g., [31, 34, 41]). In this paper the motivation to consider models with variable viscosity arises from numerical simulations of multiphase flow, which is often described by the so-called phase-field model. The phase-field approach is used to model two or more immiscible and incompressible fluids and is described by the Cahn-Hilliard (C-H) equation (originally derived in [14,16]). By taking into account the convective effect of the fluid motion, a convective form of the time-dependent C-H equation is derived (e.g., [17]).

$$\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)C = \nabla \cdot [\kappa(C)\nabla(\beta\Psi'(C) - \alpha\nabla^2 C)], \quad \text{in } \Omega \times (0, T] \tag{1.1}$$

with suitable boundary and initial conditions for the primal variable C. Here C represents the different phases and is referred to as the *phase field* or the *concentration.* It takes distinct values in each of the phases (for instance $+1$ and -1 for a binary fluid), with a smooth and rapid

* Received August 24, 2011 / Revised version received December 13, 2011 / Accepted January 16, 2012 / Published online September 24, 2012 /

change between those values in the interface zone. The coefficient $\kappa(C)$ is the so-called *mobility*, assumed to be a function of the *concentration* C. The coefficients α and β are constants. The function $\Psi(C)$ is a double-well potential, attaining its minimal value at ± 1 (under the assumption that the *concentration* C varies between $+1$ and -1). For instance, $\Psi(C) = \frac{1}{4}(C^2-1)^2$ is a common choice. For more details we refer to the classic work [46] by van der Waals and, for instance, to [12, 17, 18, 32] and the references therein. In equation (1.1), the vector $\mathbf{u}$ denotes the velocity. The term $\mathbf{u}\cdot\nabla$ represents the convective effect of the fluid motion, governed by the time-dependent incompressible Navier-Stokes (N-S) equations

$$\rho\Big(\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \mathbf{u}\cdot\nabla\mathbf{u}\Big) - \nabla\cdot(2\mu\mathbf{D}\mathbf{u}) + \nabla p = \mathbf{f} - (\beta\Psi'(C) - \alpha\nabla^2 C)\nabla C, \qquad \text{in } \Omega\times(0,T] \quad (1.2)$$

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla\cdot(\rho\mathbf{u}) = 0, \qquad \text{in } \Omega\times(0,T] \quad (1.3)$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{in } \Omega\times(0,T] \quad (1.4)$$

with some given boundary and initial conditions for $\mathbf{u}$. Here $\Omega\times(0,T]\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded, connected domain with boundary $\partial\Omega$ and $\mathbf{f}:\Omega\to\mathbb{R}^d$ is a given force field. The operator $\mathbf{D}\mathbf{u} = (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$ denotes the rate-of-strain tensor for Newtonian fluids and the force term $(\beta\Psi'(C) - \alpha\nabla^2 C)\nabla C$ denotes the surface tension force and constitutes the coupling with the C-H equation (1.1) (e.g., [21]). The coefficient μ denotes the dynamic viscosity and ρ denotes the density.

We point out that a numerical simulation of a multiphase flow problem requires to solve the coupled system, consisting of the time-dependent C-H and incompressible N-S equations, where the N-S equations are formulated in their full complexity, including the time-dependence, variable density and variable viscosity. Note, that density and viscosity remain constant within each phase, however they vary smoothly and rapidly in the interface region, which evolves with time and in space (e.g., [21]). Therefore, these can be seen as smooth functions of space and time in the whole computational domain.

In this paper we limit ourselves to the stationary incompressible N-S equations with constant density while allowing viscosity to vary. An illustrative example for such a system is a mixture of water and oil, which have the same density, however their viscosities differ much. Other examples of problems of practical importance are considered in [45], namely, extrusion with variable viscosity and a geodynamic problem with a sharp viscosity contrast (SINKER).

The main task in this paper is to analyse the effect of variable viscosity on some of the established preconditioning techniques, used for the system matrix arising form the finite element (FEM) discretization of the stationary incompressible N-S equations with constant viscosity. The structure of the paper is as follows. In Section 2 we state the problem setting. In Section 3 we recall the augmented Lagrangian (AL) method and analyse the impact of variable viscosity on the AL preconditioner. Section 4 contains numerical illustrations. Some discussion points and conclusions are stated in Section 5.

2. Problem Setting and Preliminaries

As mentioned above, in this paper we focus on preconditioners for the iterative solution of the stationary incompressible N-S problems with variable viscosity.The governing equations