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ERROR REDUCTION, CONVERGENCE AND OPTIMALITY FOR ADAPTIVE MIXED FINITE ELEMENT METHODS FOR DIFFUSION EQUATIONS*

Shaohong Du
School of Science, Chongqing Jiaotong University, Chongqing 400074, China
Email: dushhong@gmail.com
Xiaoping Xie
School of Mathematics, Sichuan University, Chengdu 610064, China
Email: xpxiec@gmail.com

Abstract

Error reduction, convergence and optimality are analyzed for adaptive mixed finite element methods (AMFEM) for diffusion equations without marking the oscillation of data. Firstly, the quasi-error, i.e. the sum of the stress variable error and the scaled error estimator, is shown to reduce with a fixed factor between two successive adaptive loops, up to an oscillation. Secondly, the convergence of AMFEM is obtained with respect to the quasi-error plus the divergence of the flux error. Finally, the quasi-optimal convergence rate is established for the total error, i.e. the stress variable error plus the data oscillation.

Mathematics subject classification: 65N30, 65N15, 65N12, 65N50.
Key words: Adaptive mixed finite element method, Error reduction, Convergence, Quasi-optimal convergence rate.

1. Introduction and Main Results

Let Ω be a bounded polygonal in $\mathbb{R}^2$. We consider the following diffusion problem with homogeneous Dirichlet boundary value:

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the diffusion tensor $A \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ is a symmetric and uniformly positive definite matrix, and $f \in L^2(\Omega)$. The choice of boundary conditions is made for ease of presentation, since similar results are valid for other boundary conditions.

Adaptive methods for the numerical solution of PDEs are now standard tools in science and engineering to achieve better accuracy with minimum degrees of freedom. The adaptive procedure of (1.1) consists of loops of the form

$$SOLVE \to ESTIMATE \to MARK \to REFINE. \tag{1.2}$$

A posteriori error estimation (ESTIMATE) is an essential ingredient of adaptivity. We refer to [1, 2, 7, 17, 30] for related works on this topic. The analysis of convergence and optimality of the whole algorithm (1.2) is still in its infancy.

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The convergence analysis of standard adaptive finite element method (AFEM) started with Döfler [16]. Döfler introduced a crucial marking, and proved the strict energy error reduction of the standard AFEM for the Laplacian under the condition that the initial mesh $\mathcal{T}_0$ satisfies a fineness assumption. Morin et al. [24, 25] showed that such strict energy error reduction can not be expected in general. Introducing the concept of data oscillation and the interior node property, they proved convergence of the standard AFEM without fineness restriction on $\mathcal{T}_0$ which is valid only for A in (1.1) being piecewise constant on $\mathcal{T}_0$. Inspired by the work by Chen and Feng [11], Mekchay and Nochetto [22] extended this result to general second elliptic operators and proved that the standard AFEM is a contraction for the total error, namely the sum of the energy error and oscillation. Recently, Cascon, et al. [10] presented a new error notion, the so-called *quasi-error*, namely the sum of the energy error and the scaled estimator, and showed without the interior node property for the self-adjoint second elliptic problem that the quasi-error is strictly reduced by the standard AFEM even though each term may not be. Very recently, in [20, 21] Hu et al. first proved the convergence of adaptive conforming and nonconforming finite element methods without marking the oscillation of data.

Besides convergence, optimality is another important issue in AFEM which was first addressed by Binev et al. [4] and further studied by Stevenson [28, 29], who showed optimality without additional coarsening required in [4, 5]. These papers [4, 5, 28, 29] are restricted to Laplace operator and rely on suitable marking by data oscillation and the interior node property. Cascon et al. [10] succeeded in establishing quasi-optimality of the AFEM without both the assumption of the interior node property and marking by data oscillation for the self-adjoint second elliptic operator. Very recently, in [20,21] Hu et al. first analyzed the optimality of adaptive conforming or nonconforming finite element methods without using the algorithm that separates the error and the reduction of data oscillation.

However, for the convergence and optimality of AMFEM, the present works are carried out only for Poisson equations: In [8], Carstensen and Hoppe proved the error reduction and convergence for only the lowest-order Raviart-Thomas element. Chen et al. [12] showed the convergence of the quasi-error and the optimality of the flux error while marking the data oscillation. In [3, 9, 18], the convergence and optimality were analyzed for only the lowest-order Raviart-Thomas element where the local refinement was performed by using simply either the estimators or the data oscillation term.

Since the approximation of the mixed finite element methods is a saddle point of the corresponding energy, there is no orthogonality available, as is one of main difficulties for the convergence and optimality of AMFEM. Since the stress variable is of interest in many applications, we especially concern the stress variable error. In this paper, our main contribution is that we develop a novel technique and show, for more general elliptic problems and more general mixed elements, the reduction property of the quasi-error (i.e., the saturation property), the convergence of the quasi-error plus the divergence of the flux error, and the quasi-optimal convergence rate of the total error with only the Dörfler Marking and without marking the oscillation.

To summarize our main results, let $\{\mathcal{T}_k, (M_k, L_k), p_k, \eta_k\}_{k\geq 0}$ be the sequence of the meshes, a pair of finite element spaces with $\mathrm{div} M_k = L_k$, the approximation solutions, the estimators produced by AMFEM in the k-th step. We prove in Section 5 that the quasi-error uniformly reduces with a fixed rate between two successive meshes, up to an oscillation of data f, namely

$$\mathcal{E}_{k+1}^2 + \gamma\eta_{k+1}^2 \leq \alpha^2(\mathcal{E}_k^2 + \gamma\eta_k^2) + C\mathrm{osc}^2(f, \mathcal{T}_k),$$