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STABILITY AND RESONANCES OF MULTISTEP COSINE METHODS*

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Abstract

In a previous paper, some particular multistep cosine methods were constructed which proved to be very efficient because of being able to integrate in a stable and explicit way linearly stiff problems of second-order in time. In the present paper, the conditions which guarantee stability for general methods of this type are given, as well as a thorough study of resonances and filtering for symmetric ones (which, in another paper, have been proved to behave very advantageously with respect to conservation of invariants in Hamiltonian wave equations). What is given here is a systematic way to analyse and treat any of the methods of this type in the mentioned aspects.

Mathematics subject classification: 35L70, 65M12, 65M99.
Key words: Exponential integrators, Multistep cosine methods, Second-order partial differential equations, Stability, Resonances.

1. Introduction

In this paper we deal with the stability and phenomenon of resonances of multistep cosine methods. These methods are designed to integrate systems of the form

$$\ddot{y}(t) = -\Omega^2 y(t) + g(t, y(t)), \tag{1.1}$$

with g a smooth function and Ω some matrix which we assume to be diagonal with real eigenvalues, in such a way that the linear part is integrated exactly. These methods have been proved to turn up very efficient when integrating the system which arises after the space discretization of a partial differential equation of second-order in time when the 'stiff' part is linear [2, 3, 5, 6, 9, 10, 13] and *High order symmetric multistep cosine methods*, by B. Cano and M. J. Moreta (unpublished). In such a way, explicit and stable methods can be obtained. We remark that the methods suggested in [5,6,9,10,13] lead to at most second-order in time under a finite-energy and non-resonance condition, but without assuming any regularity of the solution of the continuous problem. On the contrary, the methods suggested and analysed in [2, 3] and *High order symmetric multistep cosine methods* (unpublished) can lead to higher order in time without assuming any finite-energy condition but taking as a strong hypothesis enough

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regularity of the solution of the continuous problem. The study of stability (independent of the degree of stiffness of the problem) has been done for particular cases as Gautschi, MC3 and SMC4 methods in [3] but a general study is lacking in the literature yet which can be applied to methods of higher order for regular solutions, as those suggested in *High order symmetric multistep cosine methods* (unpublished). That is the first aim of the present paper.

For the sake of simplicity, we will consider explicit methods with an even number of steps, which are the ones recommended in the literature [3] and *High order symmetric multistep cosine methods* (unpublished). These methods are determined by a difference equation like

$$\rho_{h\Omega}(E)y_n = h^2\sigma_{h\Omega}(E)g(t_n, y_n), \tag{1.2}$$

where y_n approximates the exact solution $y(t_n)$, with $t_n = t_0 + nh$ (natural n), E is the operator which advances a stepsize from n to $n+1$ and

$$\rho_\epsilon(z) = z^{2k} + \alpha_{2k-1}(\epsilon)z^{2k-1} + \cdots + \alpha_0(\epsilon), \quad \sigma_\epsilon(z) = \gamma_{2k-1}(\epsilon)z^{2k-1} + \cdots + \gamma_0(\epsilon),$$

with $\{\alpha_j\}_{j=0}^{2k-1}, \{\gamma_j\}_{j=0}^{2k-1}$ certain real functions.

When the methods of this type are consistent and stable convergence follows, which implies that, for a fixed value of time, when the timestepsize diminishes, the error goes to zero (see [3] and *High order symmetric multistep cosine methods* (unpublished)). However, the values of the timestepsizes h for which clean numerical convergence of the corresponding order is observed vary a lot depending on the possible diagonal elements λ of Ω. That may lead to 'not so good' numerical results. The phenomenon which we try to avoid is called 'resonance', since it is caused by the fact that $h\lambda$ is at or near certain real distinguished values. This phenomenon has been well studied in [3] for the symmetric Gautschi and SMC4 methods for the scalar equation

$$\ddot{y}(t) = -\lambda^2 y(t) - y(t), \tag{1.3}$$

and our second aim here is to give a more general study in order to understand it, only for this equation, but for every symmetric multistep cosine method. This will allow to construct filters which avoid those resonances. We remark that the filters suggested here may not lead to uniform second-order convergence for problem (1.3), as distinct from some of the filters well discussed in [10]. However, these filters make the methods conserve its order of consistency (as high as we want) when integrating regular solutions of Hamiltonian wave or beam equations, as it is well justified in *High order symmetric multistep cosine methods* (unpublished). Remark 4.1 means to be clarifying in that sense. Besides, for space discretizations of this type of equations (much more complicated than (1.3)), we have numerically observed that resonances are also avoided in *High order symmetric multistep cosine methods* (unpublished). We also remark that symmetry of these methods is a key condition to guarantee a good behaviour with respect to conservation of invariants with time when integrating space discretizations of Hamiltonian wave equations (see [2]).

The paper is structured as follows. Section 2 deals with conditions under which stability can be assured. Section 3 gives a detailed study of resonances. Section 4 analyses how filtering must be done in order to avoid them. Finally, in Section 5 some numerical experiments are shown which corroborate previous results. For the sake of readability, the more technical proofs of the provided theorems have been written in an appendix.