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THE ULTRACONVERGENCE OF EIGENVALUES FOR BI-QUADRATIC FINITE ELEMENTS*

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Abstract

The classical eigenvalue problem of the second-order elliptic operator is approximated with bi-quadratic finite element in this paper. We construct a new superconvergent function recovery operator, from which the $O(h^8|\ln h|^2)$ ultraconvergence of eigenvalue approximation is obtained. Numerical experiments verify the theoretical results.

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Key words: Finite element method, Eigenvalue recovery, Ultraconvergence.

1. Introduction

Many post-processing techniques have been proposed for the finite element method, and they are widely used in scientific and engineering application. For the literature, readers are referred to [1, 3, 4, 29, 30, 33], and references therein.

One of the post-processing methods is the gradient recovery technique [12,14,15,32,33]. Recently, this technique has been used to improv the eigenvalue approximation by the linear finite element method [9, 17, 20]. It turned out that the convergent rate can be doubled in the most favorable situation. In this paper, we design a function recovery operator for the quadratic finite element method to enhance the eigenvalue approximation. Due to the complexity nature of the recovery operator in higher-order situation, this extension is nevertheless non-trivial as we may see from the later sections.

For the quadratic finite element method, Lin and Yang [13] discovered and proved that derivatives of bi-quartic interpolation I_4u_h of the solution of biquadratic element have 4th-order super-convergence, and the Rayleigh quotient of I_4u_h has eight-order super-convergence.

One of the strategies in eigenvalue enhancement is to use the Rayleigh quotient as e.g., in [9, 13, 18, 26]. This is also the strategy we will use in this work. The key idea is to replace the finite element gradient (or its resultants) on the numerator of the Rayleigh quotient by the recovered gradient. At the same time, the denominator has to be changed in order to consistent with the recovered gradient. Therefore, a function value recovery is also required.

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The recovery technique is most effective under uniform meshes, or the meshes with regular refinement. In this work, we only concentrate on rectangular meshes, and leave the triangular case to a forthcoming paper. The new recovery operator proposed here works remarkably well for our purpose. As we shall see in Section 5, an $O(h^8|\ln h|^2)$ convergence rate for eigenvalue approximation is obtained.

Different from the linear element case, in which the function value recovery has no superconvergence [19], the function recovery operator constructed for quadratic element in our work is superconvergent.

The paper is organized as follows. Section 2 outlines the eigenvalue problems and their Galerkin approximation. In Section 3, the derivative recovery technique is introduced, and the ultraconvergence of derivatives is shown for the bi-quadratic finite element. Then we construct the function recovery operator and discuss its properties. The eigenvalue recovery is given in the Raylaigh quotient theme in Section 4, and the main results are proved in Section 5. Finally, numerical tests are provided in Section 6 to demonstrate the effectiveness of our method.

2. Problem and Its Galerkin Approximation

Consider the following classic eigenvalue problem of the Laplacian operator:

$$\begin{cases} Lu \equiv -\nabla \cdot (\mathbf{D}\nabla u) + cu = \lambda u, \quad x \in \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

where Ω is a bounded polygonal domain, $\mathbf{D}$ is a 2×2 positive definite matrix on Ω, and c is a sufficiently smooth function.

Let $V = H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Then the variational formulation of the problem (2.1) is to seek the eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ such that

$$a(u,v) = \lambda b(u,v), \ \forall\, v \in V, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega (\mathbf{D}\nabla u)\cdot \nabla v + cuv, \quad b(u,v) = \int_\Omega uv.$$

For simplicity, in this paper we shall discuss the case when $D = I$ and $c = 0$, then the bilinear $a(\cdot,\cdot)$ is bounded and V-elliptic, namely, there exist the constants M_1 and M_2 such that

$$\begin{array}{r} |a(u,v)| \leqslant M_1\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \\ a(u,u) \geqslant M_2\|u\|_{1,\Omega}^2, \end{array} \quad \forall\, u,v \in V.$$

Let $\mathcal{T}_h$ be a quasi-uniform rectangulation of Ω. In this paper, the bi-quadratic finite element method is used to solve the problem (2.1). And let V_h denote the bi-quadratic finite element space, i.e.,

$$V_h = \Big\{ v \in C(\overline{\Omega}) : v|_\tau \in Q_2(\tau) \text{ for every rectangle element } \tau \in \mathcal{T}_h \Big\}.$$

And we denote $V_h^0 = V_h \bigcap V$. So the finite element approximation (λ_h, u_h) of (λ, u) in (2.2) can be computed as the following scheme: find $(\lambda_h, u_h) \in \mathbb{R} \times V_h^0$ such that

$$a(u_h, v) = \lambda_h(u_h, v), \ \forall\, v \in V_h^0. \tag{2.3}$$