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THE BEST L^2 NORM ERROR ESTIMATE OF LOWER ORDER FINITE ELEMENT METHODS FOR THE FOURTH ORDER PROBLEM*

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Abstract

In the paper, we analyze the L^2 norm error estimate of lower order finite element methods for the fourth order problem. We prove that the best error estimate in the L^2 norm of the finite element solution is of second order, which can not be improved generally. The main ingredients are the saturation condition established for these elements and an identity for the error in the energy norm of the finite element solution. The result holds for most of the popular lower order finite element methods in the literature including: the Powell-Sabin C^1-P_2 macro element, the nonconforming Morley element, the C^1-Q_2 macro element, the nonconforming rectangle Morley element, and the nonconforming incomplete biquadratic element. In addition, the result actually applies to the nonconforming Adini element, the nonconforming Fraeijs de Veubeke elements, and the nonconforming Wang-Xu element and the Wang-Shi-Xu element provided that the saturation condition holds for them. This result solves one long standing problem in the literature: can the L^2 norm error estimate of lower order finite element methods of the fourth order problem be two order higher than the error estimate in the energy norm?

Mathematics subject classification: 65N10, 65N15, 35J25.
Key words: L^2 norm error estimate, Energy norm error estimate, Conforming, Nonconforming, The Kirchhoff plate.

1. Introduction

We shall consider the L^2 norm error estimate of the finite element method of the Kirchhoff plate bending problem reads: Given $g \in L^2(\Omega)$ find $w \in W := H_0^2(\Omega)$ with

$$a(w, v) = (g, v)_{L^2(\Omega)} \quad \text{for all } v \in W\,. \tag{1.1}$$

The bilinear form $a(w, v)$ reads

$$a(w, v) := (\nabla^2 w, \nabla^2 v)_{L^2(\Omega)} \text{ for any } w\,, v \in W\,, \tag{1.2}$$

where $\nabla^2 w$ is the Hessian of w. For this fourth order elliptic problem, there are a number of conforming/nonconforming finite element methods in the literature, see for instance, [6, 8, 20] and the references therein.

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Let W_h be some conforming or nonconforming finite element space defined over the triangulation $\mathcal{T}_h$ of the domain $\Omega \subset \mathbb{R}^2$ into rectangles or triangles, the discrete problem reads: Find $w_h \in W_h$ such that

$$a_h(w_h, v_h) = (g, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in W_h\,. \tag{1.3}$$

The broken version $a_h(\cdot,\cdot)$ follows

$$a_h(w_h, v_h) := (\nabla_h^2 w_h, \nabla_h^2 v_h)_{L^2(\Omega)} \text{ for any } w_h\,, v_h \in W + W_h\,,$$

where ∇_h^2 is the discrete counterpart of the Hessian operator ∇^2, which is defined elementwise with respect to the triangulation $\mathcal{T}_h$ since W_h may be nonconforming. If $W_h \subset W$, we have $a_h(w_h, v_h) = a(w_h, v_h)$ for $w_h, v_h \in W_h$.

Under some continuity condition of the discrete space W_h [6, 8, 20], the discrete problem (1.3) will be well-posed and consequently admit a unique solution. Define the residual

$$\text{Res}_h(v_h) := (g, v_h)_{L^2(\Omega)} - a_h(w, v_h) \text{ for any } v_h \in W_h\,. \tag{1.4}$$

Then we have the following Strang Lemma:

$$\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)} \leq C\bigg(\sup_{v_h \in W_h} \frac{\text{Res}_h(v_h)}{\|\nabla_h^2 v_h\|_{L^2(\Omega)}} + \min_{v_h \in W_h} \|\nabla_h^2(w - v_h)\|_{L^2(\Omega)}\bigg). \tag{1.5}$$

Here and throughout this paper C is some generic positive constant which is independent of the meshsize. We are interested in some lower order methods: the nonconforming Morley element [15, 17, 21], the Powell-Sabin $C^1\!-P_2$ macro element [18], the $C^1\!-Q_2$ macro element [10], the nonconforming rectangle Morley element [24], and the nonconforming incomplete biquadratic element [16, 27]. For these discrete methods, it follows from the Strang Lemma that

$$\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)} \leq Ch\|g\|_{L^2(\Omega)}, \tag{1.6}$$

provided that $w \in H^3(\Omega) \cap H_0^2(\Omega)$. Here and throughout this paper, h denotes the meshsize which is defined by

$$h := \max_{K \in \mathcal{T}_h} h_K \text{ with } h_K \text{ the diameter of } K. \tag{1.7}$$

By the dual argument, we have

$$\|w - w_h\|_{L^2(\Omega)} + \|\nabla_h(w - w_h)\|_{L^2(\Omega)} \leq Ch^2\|g\|_{L^2(\Omega)}, \tag{1.8}$$

provided that Ω is smooth or convex, where ∇_h is the elementwise defined counterpart of the gradient operator ∇. By the approximation property of the discrete space, we have

$$\inf_{v_h \in W_h} \|w - v_h\|_{L^2(\Omega)} \leq Ch^3|w|_{H^3(\Omega)}, \tag{1.9}$$

for all the methods under consideration. Compared to the approximation result (1.9), the L^2 norm error estimate in (1.8) is obviously not optimal. Then one long standing problem for the finite element method of the fourth order problem is: can the L^2 norm error estimate of lower order finite element methods of the fourth order problem be two order higher than the error estimate in the energy norm? The aim of the paper is to prove that the L^2 norm error estimate in (1.8) can not be improved for these methods under consideration. The main ingredients are the saturation condition and the identity of the error in the energy norm.

This paper is organized as follows. In the following section, we introduce five lower order finite element methods for the fourth order problem. In Section 3, we prove that the best error estimate in the L^2 norm of the finite element solution is of second order based on the saturation condition, which will be established in Section 4. This paper ends with Section 5 where we present some conclusion and give some further comments on the other lower order finite element methods in the literature.

2. The Finite Elements of the Kirchhoff Plate Problem

This section presents some lower order finite element methods for the fourth order problem.

Suppose that the closure $\overline{\Omega}$ is covered exactly by a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ into (closed) triangles or rectangles in $2D$, that is

$$\overline{\Omega} = \cup \mathcal{T}_h \quad \text{and} \quad |K_1 \cap K_2| = 0 \quad \text{for } K_1, K_2 \in \mathcal{T}_h \quad \text{with } K_1 \neq K_2\,, \tag{2.1}$$

where $|\cdot|$ denotes the volume (as well as the length of an edge and the modulus of a vector etc, when there is no real risk of confusion). Let $\mathcal{E}$ denote the set of all edges in $\mathcal{T}_h$ with $\mathcal{E}(\Omega)$ the set of interior edges. Given any edge $E \in \mathcal{E}(\Omega)$ with length $h_E = |E|$ we assign one fixed unit normal $\nu_E := (\nu_1, \nu_2)$ and tangential vector $\tau_E := (-\nu_2, \nu_1)$. For E on the boundary we choose $\nu_E = \nu$ the unit outward normal to Ω. Once ν_E and τ_E have been fixed on E, in relation to ν_E one defines the elements $K_- \in \mathcal{T}_h$ and $K_+ \in \mathcal{T}_h$, with $E = K_+ \cap K_-$ and $\omega_E = K_+ \cup K_-$. Given $E \in \mathcal{E}(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω, with $d = 1, 2$, we denote by $[v] := (v|_{K_+})|_E - (v|_{K_-})|_E$ the jump of v across E, where $v|_{K_+}$ (resp. $v|_{K_-}$) is the restriction of v on K_+ (resp. K_-).

2.1. The Powell-Sabin $C^1 - P_2$ macro element

This is a triangle macro-element. Let M_h be some regular triangulation of the domain Ω into triangles. Then, refining each base triangle of M_h into 6, for example, connecting the center of the inscribed circle of a triangle to its three vertices and the centers of three neighboring triangles, cf. Figure 2.1, which results in the final mesh $\mathcal{T}_h$. Based on such a special triangulation, the Powell-Sabin C^1-P_2 element was created in 1977, cf. [18]. The restriction on each element $K \in \mathcal{T}_h$ of the function in the Powell-Sabin C^1-P_2 element space $W_{PS} \subset W$ is a polynomial of degree ≤ 2. The degrees of freedom are the values, and the first order derivatives on the vertexes of the macro-mesh M_h.

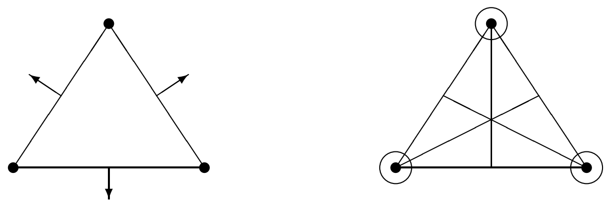

Fig. 2.1. The Morley element, and the C^1-P_2 Powell-Sabin element.

2.2. The nonconforming Morley element

This is a triangle element. The discrete space of the Morley finite element method reads [15,17,21]

$$\begin{aligned} W_M := \Big\{ v \in \mathcal{M}_{2,h}, & \int_E [\nabla v \cdot \nu_E] ds = 0 \text{ on } E \in \mathcal{E}(\Omega), \\ \text{and } & \int_E \nabla v \cdot \nu_E ds = 0 \text{ on } E \in \mathcal{E} \cap \partial\Omega \Big\}, \end{aligned} \tag{2.2}$$

where $\mathcal{M}_{2,h}$ is the space of piecewise polynomials of degree ≤ 2 over $\mathcal{T}_h$ which are continuous at all the internal nodes and vanish at all the nodes on the boundary $\partial\Omega$.

2.3. The $C^1 - Q_2$ macro element

This is a rectangle element defined over the macro-mesh. We first let M_h be a shape regular triangulation of Ω into rectangles. Then we divide each rectangle in M_h by the usual red refinement into four sub-rectangles to obtain the mesh $\mathcal{T}_h$. Let the polynomial space of separated degree k or less be

$$Q_k : = \Big\{ \sum_{0 \leq i,j \leq k} c_{ij} x^i y^j \Big\}.$$

The C^1-Q_2 macro element space is defined by [10]

$$W_{Q_2} : = \left\{ v_h \in C^1(\Omega), \, v_h|_K \in Q_2 \ \forall K \in \mathcal{T}_h, \text{ and } \ v_h\big|_{\partial\Omega} = \partial_\nu v_h\big|_{\partial\Omega} = 0 \right\}. \tag{2.3}$$

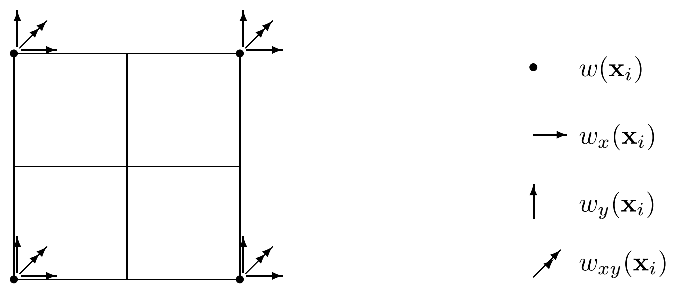

Fig. 2.2. The C^1-Q_2 macro element.

2.4. The nonconforming rectangle-Morley element

This rectangle nonconforming finite element method is proposed in [24]. The shape function space reads

$$Q_{RM}(K) : = P_2(K) + \text{span}\{x^3, y^3\}, \tag{2.4}$$

where $P_2(K)$ is the space of the polynomials of degree ≤ 2 over K. The rectangle-Morley element space is defined by

$$W_{RM} := \Big\{ v_h \in L^2(\Omega), v_h|_K \in Q_{RM}(K), \text{ the value of } v_h \text{ is continuous at all the internal vertexes, and vanishes at all the boundary vertexes , and the normal derivative is continuous at all the mid-points of the internal edges, and vanishes at all the mid-points of the boundary edges} \Big\}. \tag{2.5}$$

Since $\nabla_h v_h \cdot \nu_E$ is a linear function on all the edges, we have

$$\int_E [\nabla_h v_h] ds = 0 \text{ for any } v_h \in W_{RM} \text{ for any internal edge } E,$$

and

$$\int_E \nabla_h v_h ds = 0 \text{ for any } v_h \in W_{RM} \text{ for any boundary edge } E.$$

2.5. The nonconforming incomplete biquadratic element

This incomplete biquadratic nonconforming plate element is proposed in [27] and analyzed in [16]. The shape functions space reads

$$Q_{IB}(K) := P_2(K) + \text{span}\{x^2 y, y^2 x\}. \tag{2.6}$$

The incomplete biquadratic element space is defined by

$$W_{IB} := \{ v_h \in L^2(\Omega), v_h|_K \in Q_{IB}(K), \text{ the value of } v_h \text{ is continuous at all the internal vertexes, and vanishes at all the boundary vertexes , and the normal derivative is continuous at all the mid-points of the internal edges, and vanishes at all the mid-points of the boundary edges} \}. \tag{2.7}$$

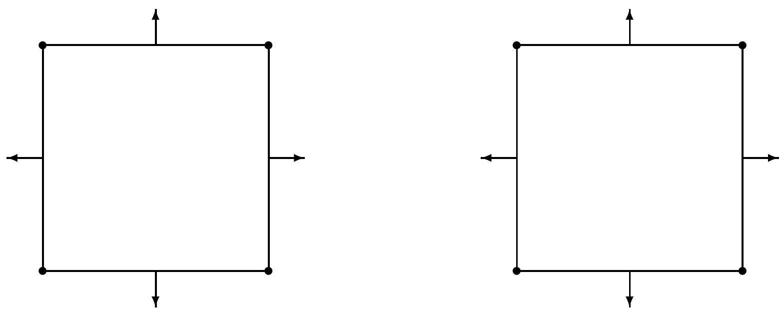

Fig. 2.3. The nonconforming rectangle-Morley element, and the nonconforming incomplete biquadratic element.

3. The Best L^2 Norm Error Estimate

In this section, we shall prove that the L^2 norm error estimate is of second order and that this estimate can not be improved. For the analysis, we need the following saturation condition:

Lemma 3.1. *Let $w \in H^3(\Omega) \cap H_0^2(\Omega)$ and w_h be the solutions to the problems* (1.1) *and* (1.3)*, respectively. Then*

$$\beta h \leq \|\nabla_h^2(w - w_h)\|_{L^2(\Omega)}, \tag{3.1}$$

with a positive constant β which is independent of the meshsize for all the finite element methods described in the previous section provided that the mesh size h is small enough.

We shall postpone the proof of this lemma to the next section. With the solutions w of the continuous problem and w_h of the discrete problem, we have the following identity

$$\begin{aligned} &\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)}^2 = a_h(w - w_h, w - w_h) \\ =&(g, w - w_h)_{L^2(\Omega)} + 2((g, w_h) - a_h(w_h, w)). \end{aligned} \tag{3.2}$$

It follows from this identity and the saturation condition (3.1) that the following theorem holds

Theorem 3.1. *Let $w \in H^3(\Omega) \cap H_0^2(\Omega)$ be the solution of the problem* (1.1) *and w_h be the solution of the problem* (1.3) *by the Powell-Sabin $C^1 - P_2$ macro element and the $C^1 - Q_2$ macro element from the previous section. There exists a positive constant α independent of the meshsize h such that*

$$\alpha h^2 \leq \|w - w_h\|_{L^2(\Omega)} \tag{3.3}$$

when the meshsize h is sufficiently small.

Proof. The identity (3.2) will become

$$\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)}^2 = (g, w - w_h)_{L^2(\Omega)} \tag{3.4}$$

for the conforming finite element method. Assume that the lower bound (3.3) is not true. Then for arbitrary $\epsilon > 0$ there exists sufficiently small h such that

$$\frac{\|w - w_h\|_{L^2(\Omega)}}{h^2} \leq \epsilon^2. \tag{3.5}$$

It follows from (3.4) and the Cauchy-Schwarz inequality that

$$\frac{\|\nabla_h^2(w - w_h)\|_{L^2(\Omega)}}{h} \leq C\epsilon, \tag{3.6}$$

which contradicts with the saturation condition (3.1). □

To analyze the nonconforming Morley element, we need the canonical interpolation operator $\Pi_M : W \to W_M$ defined by

$$\begin{aligned} &(\Pi_M)v(p) = v(p) \text{ for any node } p \text{ of } \mathcal{T}_h, \\ &\int_E \frac{\partial \Pi_M v}{\partial \nu_E} ds = \int_E \frac{\partial v}{\partial \nu_E} ds \text{ for any edge } E \text{ of } \mathcal{T}_h, \end{aligned} \tag{3.7}$$

for any $v \in W$. We have the following properties for this interpolation [11, 15, 20, 21]

$$(\nabla_h^2 s_h, \nabla_h^2 (I - \Pi_M) v)_{L^2(\Omega)} = 0 \qquad \text{for any } s_h \in W_M \text{ and } v \in W, \tag{3.8a}$$
$$\|(I - \Pi_M) v\|_{L^2(\Omega)} \le C h^3 |v|_{H^3(\Omega)} \quad \text{for any } v \in W \cap H^3(\Omega). \tag{3.8b}$$

Let $\Pi_{RM} : W \to W_{RM}$ be the Galerkin projection operator defined by

$$a_h(v - \Pi_{RM} v, s_h) = 0 \text{ for any } s_h \in W_{RM}, \tag{3.9}$$

for any $v \in W$. Let $\Pi_{IB} : W \to W_{IB}$ be the Galerkin projection operator defined by

$$a_h(v - \Pi_{IB} v, s_h) = 0 \text{ for any } s_h \in W_{IB}, \tag{3.10}$$

for any $v \in W$.

Theorem 3.2. *Let $w \in H^3(\Omega) \cap H_0^2(\Omega)$ be the solution of the problem* (1.1) *and w_h be the solution of the problem* (1.3) *by the nonconforming Morley element* (2.2)*, the nonconforming rectangle-Morley element* (2.5)*, and the nonconforming incomplete biquadratic element* (2.7)*, respectively. There exists a positive constant α independent of the meshsize h such that*

$$\alpha h^2 \le \|w - w_h\|_{L^2(\Omega)} + \|w - \Pi_h w\|_{L^2(\Omega)}, \tag{3.11}$$

when the meshsize h is small enough. Here $\Pi_h = \Pi_M, \Pi_{RM}, \Pi_{IB}$, respectively.

Proof. It follows from the discrete problem (1.3), the identity (3.2) and the definition of Π_h that

$$\|\nabla_h^2 (w - w_h)\|_{L^2(\Omega)}^2 = (g, w - w_h)_{L^2(\Omega)} + 2(g, w_h - \Pi_h w). \tag{3.12}$$

In the case $\alpha h^2 \le \|w - \Pi_h w\|_{L^2(\Omega)}$, we have already gotten the desired result. On the other side, we can follow the same line for the proof of (3.3) to obtain that $\alpha h^2 \le \|w - w_h\|_{L^2(\Omega)}$. □

Remark 3.1. For the Morley element, this theorem and the estimate in (3.8b) show that

$$\alpha h^2 \le \|w - w_h\|_{L^2(\Omega)}, \tag{3.13}$$

when the meshsize h is sufficiently small.

4. The Saturation Condition

Let $w \in H^3(\Omega) \cap H_0^2(\Omega)$ be the solution of the fourth order elliptic problem. Let W_h be some lower order conforming or nonconforming approximation space to $H^2(\Omega)$ over the mesh $\mathcal{T}_h$ in the following sense:

$$\sup_{v \in H^3(\Omega) \cap H_0^2(\Omega)} \inf_{v_h \in W_h} \|\nabla_h^2 (v - v_h)\|_{L^2(\Omega)} \le C h |v|_{H^3}. \tag{4.1}$$

In the following we let $\nabla^\ell v$ denote the ℓ-th order tensor of all ℓ-th order derivatives of v, for instance, $\ell = 1$ the gradient, and $\ell = 2$ the Hessian matrix, and that ∇_h^ℓ are the piecewise counterparts of ∇^ℓ defined element by element.

The following four conditions are sufficient for the saturation condition.

H1. There exists a local interpolation operator Π from the space $H^3(\Omega)$ to some higher order finite element space than the finite element space W_h under consideration;

H2. The following Poincare inequality

$$\|\nabla_h^2(w-\Pi w)\|_{L^2(\Omega)} \leq Ch\|\nabla_h^3(w-\Pi w)\|_{L^2(\Omega)}, \tag{4.2}$$

holds for the local interpolation operator Π;

H3. The following basic approximation property

$$\|\nabla_h^3(w-\Pi w)\|_{L^2(\Omega)} \to 0 \text{ when } h \to 0, \tag{4.3}$$

holds for the local interpolation operator Π;

H4. At least one fixed component of $\nabla_h^3 v_h$ vanishes for all $v_h \in W_h$ and while the L^2 norm of the same component of $\nabla^3 w$ is nonzero.

Theorem 4.1. *Suppose conditions H1-H4 hold for the discrete space W_h and the exact solution $w \in H^3(\Omega) \cap H_0^2(\Omega)$. Then,*

$$\beta h \leq \|\nabla_h^2(w-w_h)\|_{L^2(\Omega)} \text{ with a positive constant } \beta, \tag{4.4}$$

when the mesh size h is small enough.

Proof. The detailed proof for the more general case can be found in [9, Theorem A.1]. For the readers' convenience, we only sketch the proof for the case under consideration. By the condition H4, we let $\mathfrak{N}$ denote the multi-index set of $\kappa = (\kappa_1, \kappa_2)$ such that $|\kappa| = \kappa_1 + \kappa_2 = 3$ and that

$$\frac{\partial^{|\kappa|} v_h|_K}{\partial x^{\kappa_1} \partial y^{\kappa_2}} \equiv 0 \text{ for any } K \in \mathcal{T}_h \text{ and } v_h \in W_h \text{ while } \left\|\frac{\partial^{|\kappa|} w}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(\Omega)} \neq 0. \tag{4.5}$$

Hence it follows from the triangle inequality and the piecewise inverse estimate that

$$\begin{aligned}
\sum_{\kappa\in\mathfrak{N}} \left\|\frac{\partial^{|\kappa|} w}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(\Omega)}^2 &= \sum_{\kappa\in\mathfrak{N}} \sum_{K\in\mathcal{T}_h} \left\|\frac{\partial^{|\kappa|} (w-w_h)}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(K)}^2 \\
\leq & 2 \sum_{\kappa\in\mathfrak{N}} \sum_{K\in\mathcal{T}_h} \left(\left\|\frac{\partial^{|\kappa|} (w-\Pi w)}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(K)}^2 + \left\|\frac{\partial^{|\kappa|} (\Pi w-w_h)}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(K)}^2 \right) \\
\leq & C\big(\|\nabla_h^3(w-\Pi w)\|_{L^2(\Omega)}^2 + h^{-2}\|\nabla_h^2(\Pi w-w_h)\|_{L^2(\Omega)}^2\big).
\end{aligned} \tag{4.6}$$

By the Poincare inequality in the condition H2 and the triangle inequality, it follows

$$\sum_{\kappa\in\mathfrak{N}} \left\|\frac{\partial^{|\kappa|} w}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(\Omega)}^2 \leq C\big(\|\nabla_h^3(w-\Pi w)\|_{L^2(\Omega)}^2 + h^{-2}\|\nabla_h^2(w-w_h)\|_{L^2(\Omega)}^2\big). \tag{4.7}$$

Finally it follows from the condition H3 that

$$h^2 \sum_{\kappa\in\mathfrak{N}} \left\|\frac{\partial^{|\kappa|} w}{\partial x^{\kappa_1} \partial y^{\kappa_2}}\right\|_{L^2(\Omega)}^2 \leq C\|\nabla_h^2(w-w_h)\|_{L^2(\Omega)}^2, \tag{4.8}$$

when the meshsize is sufficiently small, which completes the proof. $\square$

Next we shall use the above theorem to prove the saturation condition for all the finite element methods under consideration. The main tasks are to construct the local interpolation operator Π and check the conditions H1-H4. For the Morley element, such an operator was constructed in [9]. The conditions H1-H4 were checked therein. The same argument actually applies to the conforming Powell-Sabin element. However, the argument therein can not be extended to the other elements considered herein. In this section, we shall give a systematic construction of such an operator.

Given any element K, define $\Pi_K v \in P_3(K)$ by

$$\int_K \nabla^\ell \Pi_K v dxdy = \int_K \nabla^\ell v dxdy, \ell = 0, 1, 2, 3, \tag{4.9}$$

for any $v \in H^3(K)$. Note that the operator Π_K is well-posed. Since $\int_K \nabla^2(v - \Pi_K v)dxdy = 0$,

$$\|\nabla^2(v - \Pi_K v)\|_{L^2(K)} \leq Ch_K \|\nabla^3(v - \Pi_K v)\|_{L^2(K)}. \tag{4.10}$$

Finally, define the global operator Π by

$$\Pi|_K = \Pi_K \text{ for any } K \in \mathcal{T}_h. \tag{4.11}$$

This proves the conditions H1 and H2 for all the elements herein. It follows from the very definition of Π_K in (4.9) that

$$\nabla_h^3 \Pi v = \Pi_0 \nabla^3 v \tag{4.12}$$

with Π_0 the L^2 piecewise constant projection operator with respect to $\mathcal{T}_h$. Since the piecewise constant functions are dense in the space $L^2(\Omega)$,

$$\|\nabla_h^3(v - \Pi v)\|_{L^2(\Omega)} \to 0 \text{ when } h \to 0, \tag{4.13}$$

which proves the condition H3. It remains to show the condition H4.

1. For the conforming Powell-Sabin $C^1 - P_2$ macro-element, and the nonconforming Morley element, it holds that $\nabla_h^3 w_h \equiv 0$ for all discrete functions w_h of these methods. Since $w \in W$, there exists at least one component of $\nabla^3 w$ which is nonzero. This establishes the condition H4 for these two methods.

2. For any $v_h \in W_{Q_2}$, we have $\frac{\partial^3 v_h}{\partial x^3} = \frac{\partial^3 v_h}{\partial y^3} \equiv 0$. Since $w \in W$, we have $|\frac{\partial^3 w}{\partial x^3}| + |\frac{\partial^3 w}{\partial y^3}| \neq 0$. This proves H4 for this element. A similar argument applies to the nonconforming rectangle-Morley element and the nonconforming incomplete biquadratic element.

5. Conclusions and Comments

In this paper we analyze the L^2 norm error estimate for five lower order finite element methods for the fourth order problem and prove that it has the same convergence rate as the H^1 norm error estimate. The analysis equally applies to other lower order methods, for instance, the Adini element [13], the Fraeijs de Veubeke elements [13], and the Wang-Xu element [23] and the Wang-Shi-Xu element [25] provided that the saturation condition (3.1) holds for them. Note that for the Adini element and the second Fraeijs de Veubeke element the saturation condition (3.1) implies that the consistency error in the Strang Lemma dominates the approximate error. For this case, the estimate (3.13) holds for them.

Some remarks on the conclusions are in order.

Remark 5.1. When the domain is smooth, we have the following regularity [5]

$$\|w\|_{H^4(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{5.1}$$

This implies that the L^2 norm error estimate of the higher order methods can be two order higher than the error estimate in the energy norm for the smooth domain, see [6, 8]. When Ω is a convex polygonal domain with interior angles $\omega_1,\cdots,\omega_\ell$ and $\omega := \max_{i\leq\ell}\omega_i$. Let $s_0 := \min\{\mathrm{Re}(Z)|\sin^2(Z_i\omega) = Z_i^2\sin^2(\omega),\ \ i = 1,\cdots,\ell\}$. It follows from the shift theorem due to [2] that

$$\|w\|_{H^{2+s_0-\epsilon}(\Omega)} \leq C\|f\|_{L^2(\Omega)} \text{ for any } \epsilon > 0. \tag{5.2}$$

In particular, when $0 < \omega \leq 0.7\pi$, we have $s_0 = 2$. Therefore, a similar dual argument in [6, 8] proves that the L^2 norm error estimate of the higher order methods can be $s_0 - \epsilon$ order higher than the error estimate in the energy norm.

As a summary, the L^2 norm error estimate for finite element methods of the fourth order problem is almost done.

Remark 5.2. The argument in this paper can be used to establish the lower bound of the error estimates of the Morley-Wang-Xu elements for the $2m$-th order problem [22]. In fact, we have

$$\alpha h^2 \leq \|\nabla_h^\ell(w - w_h)\|_{L^2(\Omega)}, \ell = 0,\cdots,m-1, \tag{5.3}$$

where w and w_h are the solutions of the continuous and discrete problems, respectively.

Remark 5.3. Let $e \in H_0^1(\Omega)$ be errors of the conforming finite element methods for the second order elliptic problems. Suppose that the following error estimate holds, namely,

$$\|e\|_{H^1(\Omega)} \leq Ch^m\|u\|_{H^{m+1}(\Omega)} \text{ with the exact solution } u, \tag{5.4}$$

then for $m \geq 2$ it holds that

$$\|e\|_{H^{1-m}(\Omega)} := \sup_{v\in H^{m-1}(\Omega)} \frac{(e, v)_{L^2(\Omega)}}{\|v\|_{H^{m-1}(\Omega)}} \leq Ch^{2m}\|u\|_{H^{m+2}(\Omega)}. \tag{5.5}$$

Such a kind of estimates are frequently used in the analysis for superconvergence of the finite element methods, see, for instance, [7, 14] and [28]. It is usually assumed in the literature that the above estimate can not be improved generally, namely,

$$\alpha h^{2m} \leq \|e\|_{H^{1-m}(\Omega)} \tag{5.6}$$

for some positive constant α. However, the rigorous proof is missed in the literature. We point out that a similar argument herein can actually prove the above lower bound estimate provided that we have the following saturation condition

$$\beta h^m \leq \|e\|_{H^1(\Omega)}. \tag{5.7}$$

Generally, it holds

$$\alpha h^{2m} \leq \|e\|_{H^{1-\ell}(\Omega)} \text{ for all } \ell \geq m \geq 1 \text{ and some positive constant } \alpha. \tag{5.8}$$

Note that the saturation condition (5.7) holds for most of finite element methods in the literature. To this end, the readers only need to follow (4.9) to define a local interpolation operator Π and then check the conditions in H1-H4.

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