Journal of Computational Mathematics
Vol.30, No.5, 2012, 461–482.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1201-m3848

PRECONDITIONING THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VARIABLE VISCOSITY*

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Abstract

This paper deals with preconditioners for the iterative solution of the discrete Oseen problem with variable viscosity. The motivation of this work originates from numerical simulations of multiphase flow, governed by the coupled Cahn-Hilliard and incompressible Navier-Stokes equations. The impact of variable viscosity on some known preconditioning technique is analyzed. Theoretical considerations and numerical experiments show that some broadly used preconditioning techniques for the Oseen problem with constant viscosity are also efficient when the viscosity is varying.

Mathematics subject classification: 65F10, 65F08, 65N30.
Key words: Navier-Stokes equations, Saddle point systems, Augmented Lagrangian, Finite elements, Iterative methods, Preconditioning.

1. Introduction

In this paper we consider preconditioned iterative solution methods for the stationary incompressible Navier-Stokes (N-S) equations with variable viscosity. Here we assume that the kinematic viscosity coefficient is a smooth function, such that

$$0 < \nu_{\min} \leq \nu(\mathbf{x}) \leq \nu_{\max},$$

where $\nu_{\min}$ and $\nu_{\max}$ denote its minimal and maximal value. Many mathematical models in fluid dynamics involve non-constant viscosity. For example, viscosity is a function of the temperature in convection flows (e.g., [19, 44]); it is a function of pressure and the rate-of-strain tensor in non-Newtonian flows (e.g., [11, 42]). In some quasi-Newtonian flows the variable viscosity may also depend on pressure and shear (e.g., [31, 34, 41]). In this paper the motivation to consider models with variable viscosity arises from numerical simulations of multiphase flow, which is often described by the so-called phase-field model. The phase-field approach is used to model two or more immiscible and incompressible fluids and is described by the Cahn-Hilliard (C-H) equation (originally derived in [14,16]). By taking into account the convective effect of the fluid motion, a convective form of the time-dependent C-H equation is derived (e.g., [17]).

$$\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)C = \nabla \cdot [\kappa(C)\nabla(\beta\Psi'(C) - \alpha\nabla^2 C)], \quad \text{in } \Omega \times (0, T] \tag{1.1}$$

with suitable boundary and initial conditions for the primal variable C. Here C represents the different phases and is referred to as the *phase field* or the *concentration*. It takes distinct values in each of the phases (for instance $+1$ and -1 for a binary fluid), with a smooth and rapid

* Received August 24, 2011 / Revised version received December 13, 2011 / Accepted January 16, 2012 / Published online September 24, 2012 /

change between those values in the interface zone. The coefficient $\kappa(C)$ is the so-called *mobility*, assumed to be a function of the *concentration* C. The coefficients α and β are constants. The function $\Psi(C)$ is a double-well potential, attaining its minimal value at ± 1 (under the assumption that the *concentration* C varies between $+1$ and -1). For instance, $\Psi(C) = \frac{1}{4}(C^2-1)^2$ is a common choice. For more details we refer to the classic work [46] by van der Waals and, for instance, to [12, 17, 18, 32] and the references therein. In equation (1.1), the vector $\mathbf{u}$ denotes the velocity. The term $\mathbf{u}\cdot\nabla$ represents the convective effect of the fluid motion, governed by the time-dependent incompressible Navier-Stokes (N-S) equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot(2\mu\mathbf{D}\mathbf{u}) + \nabla p = \mathbf{f} - (\beta\Psi'(C) - \alpha\nabla^2 C)\nabla C, \qquad \text{in } \Omega\times(0,T] \quad (1.2)$$

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla\cdot(\rho\mathbf{u}) = 0, \qquad \text{in } \Omega\times(0,T] \quad (1.3)$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{in } \Omega\times(0,T] \quad (1.4)$$

with some given boundary and initial conditions for $\mathbf{u}$. Here $\Omega\times(0,T]\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded, connected domain with boundary $\partial\Omega$ and $\mathbf{f}:\Omega\to\mathbb{R}^d$ is a given force field. The operator $\mathbf{D}\mathbf{u} = (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$ denotes the rate-of-strain tensor for Newtonian fluids and the force term $(\beta\Psi'(C) - \alpha\nabla^2 C)\nabla C$ denotes the surface tension force and constitutes the coupling with the C-H equation (1.1) (e.g., [21]). The coefficient μ denotes the dynamic viscosity and ρ denotes the density.

We point out that a numerical simulation of a multiphase flow problem requires to solve the coupled system, consisting of the time-dependent C-H and incompressible N-S equations, where the N-S equations are formulated in their full complexity, including the time-dependence, variable density and variable viscosity. Note, that density and viscosity remain constant within each phase, however they vary smoothly and rapidly in the interface region, which evolves with time and in space (e.g., [21]). Therefore, these can be seen as smooth functions of space and time in the whole computational domain.

In this paper we limit ourselves to the stationary incompressible N-S equations with constant density while allowing viscosity to vary. An illustrative example for such a system is a mixture of water and oil, which have the same density, however their viscosities differ much. Other examples of problems of practical importance are considered in [45], namely, extrusion with variable viscosity and a geodynamic problem with a sharp viscosity contrast (SINKER).

The main task in this paper is to analyse the effect of variable viscosity on some of the established preconditioning techniques, used for the system matrix arising form the finite element (FEM) discretization of the stationary incompressible N-S equations with constant viscosity. The structure of the paper is as follows. In Section 2 we state the problem setting. In Section 3 we recall the augmented Lagrangian (AL) method and analyse the impact of variable viscosity on the AL preconditioner. Section 4 contains numerical illustrations. Some discussion points and conclusions are stated in Section 5.

2. Problem Setting and Preliminaries

As mentioned above, in this paper we focus on preconditioners for the iterative solution of the stationary incompressible N-S problems with variable viscosity.The governing equations

read as follows:

$$\begin{aligned} -\nabla\cdot(2\nu(\mathbf{x})\mathbf{Du})+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p&=\mathbf{f}, && \text{in } \Omega\\ \nabla\cdot\mathbf{u}&=0, && \text{in } \Omega \end{aligned} \tag{2.1}$$

where the coefficient $\nu(\mathbf{x})$ denotes the kinematic viscosity, defined as $\nu=\frac{\mu}{\rho}$, and for simplicity, we only consider Dirichlet boundary conditions in this paper.

Remark 2.1. (1) In the case of constant viscosity and with the condition $\nabla\cdot\mathbf{u}=0$, we have

$$\begin{aligned} \nabla\cdot(2\nu\mathbf{Du}) =&\nabla\cdot(\nu\nabla\mathbf{u})+\nabla\cdot(\nu\nabla^T\mathbf{u})\\ =&\nu\nabla^2\mathbf{u}+\nu\nabla(\nabla\cdot\mathbf{u})\\ =&\nu\nabla^2\mathbf{u}. \end{aligned}$$

(2) In non-Newtonian flows, since ν also depends on $\mathbf{u}$, two of the terms in (2.1) may exhibit nonlinear behavior: $\nabla\cdot(2\nu(\mathbf{x})\mathbf{Du})$ and $(\mathbf{u}\cdot\nabla)\mathbf{u}$. In various related works, the nonlinear convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is handled in different ways. In [26], for example, where $\nu=\nu(\mathbf{u})$, at the nth nonlinear iteration the term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is moved to the right hand side and evaluated by the previously computed approximation for $\mathbf{u}$, i.e., $\widetilde{\mathbf{f}}=\mathbf{f}-(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1}$. Then one solves a nonlinear Stokes-type problem with variable viscosity and the matrix to be solved is symmetric. For that formulation, a block-preconditioner involving a mass-type matrix for the pressure is proposed and analyzed in [26]. However, the above treatment of the nonlinear convection term may be unsuitable for strongly convection-dominated problems, i.e., when the viscosity is very small (see, e.g., the numerical experiments in [26]).

Another approach is to linearize the convection term and incorporate it into the system matrix. Then, during each nonlinear iteration, an Oseen-type problem with variable viscosity has to be solved and the system matrix is nonsymmetric. In this paper, we focus on the latter technique.

Here, when solving (2.1) we choose to apply Picard iterations. The technique requires to solve a sequence of approximate solutions of the linear Oseen problem (e.g., [23]), which reads as follows:
At each Picard iteration, find $\mathbf{u}:\Omega\to\mathbb{R}^d$ and $p:\Omega\to\mathbb{R}$ satisfying

$$\begin{aligned} -\nabla\cdot(2\nu(\mathbf{x})\mathbf{Du})+(\mathbf{w}\cdot\nabla)\mathbf{u}+\nabla p&=\mathbf{f}, && \text{in } \Omega\\ \nabla\cdot\mathbf{u}&=0, && \text{in } \Omega \end{aligned} \tag{2.2}$$

subject to suitable boundary conditions for $\mathbf{u}$ on $\partial\Omega$, in this paper $\mathbf{u}=\mathbf{u}_b$. Here $\mathbf{w}=\mathbf{u}^{(k-1)}$ is the velocity, which has been computed in the previous Picard iteration, and is updated at every nonlinear step.

Let $\mathbf{H}^1_{E_0}=\{\mathbf{v}\in\mathcal{H}^1(\Omega)|\ \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\}$. The weak formulation of (2.2) reads as follows: Find $\mathbf{u}\in\mathbf{H}^1_{E_0}$ and $\mathbf{p}\in L_2(\Omega)$ such that

$$\begin{aligned} &(2\nu(\mathbf{x})\mathbf{Du},\mathbf{Dv})+((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v})-(\nabla\cdot\mathbf{v},\mathbf{p})=(\mathbf{f},\mathbf{v}),\\ &(\nabla\cdot\mathbf{u},\mathbf{q})=0, \end{aligned} \tag{2.3}$$

for all $\mathbf{v}\in\mathbf{H}^1_{E_0}$ and all $\mathbf{q}\in L_2(\Omega)$.

Let $\mathbf{X}_{E_0}^h$ and P^h be finite dimensional subspaces of $\mathbf{H}_{E_0}^1$ and $L_2(\Omega)$, and let $\{\vec{\varphi}_i\}_{1\le i\le n_u}$ be the nodal basis of $\mathbf{X}_{E_0}^h$ and $\{\phi_i\}_{1\le i\le n_p}$ be the nodal basis of P^h such that,

$$\mathbf{u}_h = \sum_{i=1}^{n_u} \mathbf{u}_i\vec{\varphi}_i, \quad \mathbf{p}_h = \sum_{i=1}^{n_p} p_i\phi_i,$$

where n_u and n_p are the total number of unknowns for the velocity and pressure. The linear systems arising from the weak formulation (2.3) are of the form

$$\begin{bmatrix} F & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \mathcal{A}\mathbf{x} = \mathbf{b}, \tag{2.4}$$

where the system matrix $\mathcal{A} = \begin{bmatrix} F & B^T \\ B & O \end{bmatrix}$ is nonsymmetric of saddle point form. We also assume that the discretization is done using a stable pair of FEM spaces, satisfying the Ladyzenskaya-Babuška-Brezzi (LBB) condition (e.g., [23]). The unknown vector $\mathbf{u}_h$ is the discrete velocity vector and $\mathbf{p}_h$ is the discrete pressure vector. Combining them together we set $\mathbf{x}^T = [\mathbf{u}_h^T \;\; \mathbf{p}_h^T]$. The matrix $B \in \mathbb{R}^{n_u\times n_p}$ corresponds to the discrete (negative) divergence operator and B^T corresponds to the discrete gradient operator (e.g., [23]). Clearly, when considering variable viscosity, the difference, compared to the stationary incompressible N-S equations with constant viscosity, can be observed in the pivot block $F \in \mathbb{R}^{n_u\times n_u}$, which, in the case of variable viscosity, has the form $F = A_\nu + N$. We assume that F is nonsingular and discuss A_ν and N below.

Based on the weak formulation (2.3) and the nodal basis $\{\vec{\varphi}_i\}_{1\le i\le n_u}$, the matrix A_ν is the discrete operator, corresponding to the term $(2\nu(\mathbf{x})\mathbf{D}\mathbf{u}, \mathbf{D}\mathbf{v})$, i.e.,

$$A_\nu \in \mathbb{R}^{n_u\times n_u}, \qquad [A_\nu]_{i,j} = (2\nu(\mathbf{x})\mathbf{D}\vec{\varphi}_i, \mathbf{D}\vec{\varphi}_j). \tag{2.5}$$

Thus, the matrix A_ν, defined in (2.5), is symmetric and positive definite (see more details in [29]). The matrix N is the discrete operator, corresponding to the term $((\mathbf{w}\cdot\nabla\mathbf{u}), \mathbf{v})$, i.e.,

$$N \in \mathbb{R}^{n_u\times n_u}, \qquad [N]_{i,j} = ((\mathbf{w}\cdot\nabla\vec{\varphi}_j), \vec{\varphi}_i). \tag{2.6}$$

For the purpose of the analysis in Section 3, we consider the following modified form of the weak formulation (2.3)

$$\begin{aligned} &(2\nu(\mathbf{x})\mathbf{D}\mathbf{u}, \mathbf{D}\mathbf{v}) + \frac{1}{2}(((\mathbf{w}\cdot\nabla)\mathbf{u}, \mathbf{v}) - ((\mathbf{w}\cdot\nabla)\mathbf{v}, \mathbf{u})) - (\nabla\cdot\mathbf{v}, \mathbf{p}) = (\mathbf{f}, \mathbf{v}), \\ &(\nabla\cdot\mathbf{u}, \mathbf{q}) = 0. \end{aligned} \tag{2.7}$$

This formulation is discussed in [27]. The difference between formulations (2.3) and (2.7) exhibits itself in the matrix, corresponding to the convection part. Let $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \frac{1}{2}(c(\mathbf{w}, \mathbf{u}, \mathbf{v}) - c(\mathbf{w}, \mathbf{v}, \mathbf{u}))$, where $c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = ((\mathbf{w}\cdot\nabla)\mathbf{u}, \mathbf{v})$. It is straightforward to see that $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = -\tilde{c}(\mathbf{w}, \mathbf{v}, \mathbf{u})$ for any $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}_{E_0}^1$. Thus, the matrix N_s, corresponding to the discrete operator $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v})$, is skew-symmetric, and $N_s = \frac{1}{2}(N - N^T)$ (N is the same as in (2.6)).

Consider the matrices A_ν and N_s, also assume that A_ν is nonsingular. Then the modulus of the eigenvalues of the matrix $\widetilde{N} = A_\nu^{-1/2} N_s A_\nu^{-1/2}$ is bounded independently of the mesh size h. The proof can be found in [29]. For completeness, we include it as an appendix. This result is used in [22] without a rigorous proof. Here it is applied in Section 3 where formulation (2.7) is used to derive some spectrum bounds. The numerical experiments in Section 4 are based on the original formulation (2.3).

3. Preconditioning Strategy

Now we turn to the task of constructing an efficient preconditioner for the system matrix $\mathcal{A}$ in (2.4), which is nonsymmetric of saddle point form. Preconditioned iterative solution methods for saddle point problems have been studied intensively during the last 30 years (e.g., [5, 7, 23] and the references therein). The exact factorization of a general matrix of two-by-two block form is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}, \tag{3.1}$$

where I_1 and I_2 are identity matrices of proper dimensions. The pivot block A_{11} is assumed to be nonsingular and $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the exact Schur complement matrix. In our case, $A_{11} = F$, $A_{12} = B^T$, $A_{21} = B$ and $A_{22} = O$. So, $S = -BF^{-1}B^T$.

Some of the best known preconditioners for such two-by-two block matrices are based on the exact factorization (3.1), where some of the blocks are approximated or neglected. Most often, block lower- or upper-triangular preconditioners are used, of the form

$$\mathcal{M}_L = \begin{bmatrix} \widetilde{A}_{11} & O \\ A_{21} & \widetilde{S} \end{bmatrix}, \qquad \mathcal{M}_U = \begin{bmatrix} \widetilde{A}_{11} & A_{12} \\ 0 & \widetilde{S} \end{bmatrix}. \tag{3.2}$$

Here, $\widetilde{A}_{11}$ denotes some approximation of A_{11}, given on explicit form or implicitly defined via iterative solution methods. The matrix $\widetilde{S}$ is some approximation of the exact Schur complement S.

The quality of the preconditioners in (3.2) depends on how well the pivot block A_{11} and the exact Schur complement matrix S are approximated, see [4] for a recent analysis. Compared to the approximations of A_{11}, the most challenging task, however, is how to construct numerically and computationally efficient approximations for the Schur complement matrix S, which is in general dense and not cheap to be formed explicitly. The research on Schur complement approximations for the Oseen problem with constant viscosity has been an active field of research during the past years (e.g., [22, 23, 40] and the references therein, also [36]). Since the system matrices, arising from the Oseen problem, share the same structure for both constant and variable viscosity, except for the difference in the pivot block, it is a natural idea to extend the known approximation techniques to the Oseen problem with variable viscosity. For example, a scaled pressure mass matrix $M_\nu = \{M_{\nu ij}\} \in \mathbb{R}^{n_p \times n_p}$ with $[M_\nu]_{ij} = (\nu^{-1}\phi_i, \phi_j)$ is proposed in [26] as an approximation of the negative Schur complement of the system matrix.

In this paper we utilize the same AL preconditioning strategy as shown in [8, 9, 28] for the Oseen problem with constant viscosity and extend it to the variable viscosity case. Following the standard AL technique (for an earlier reference, see [24]), we first algebraically transform the system (2.4) into an equivalent one

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \mathcal{A}_\gamma \mathbf{x} = \widehat{\mathbf{b}}, \tag{3.3}$$

where $\widehat{\mathbf{f}} = \mathbf{f} + \gamma B^T W^{-1}\,\mathbf{g}$, and $\gamma > 0$ and W are suitable scalar and matrix parameters. Clearly, the transformation (3.3) does not change the solution for any value of γ and any nonsingular matrix W.

The equivalent system (3.3) is what we intend to solve and the extension of some known efficient preconditioner to the system matrix $\mathcal{A}_\gamma$ in the case of variable viscosity is the main

issue considered in this section. Here, we again choose the AL-type preconditioner, which is originally proposed in [8] and used in [9, 28] for the Oseen problem with constant viscosity.

The AL-type preconditioner for the transformed system matrix $\mathcal{A}_\gamma$ in (3.3) is of the form

$$\mathcal{M}_L = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}. \tag{3.4}$$

It is not difficult to see that the exact Schur complement $S_{\mathcal{A}_\gamma} = -B(F+\gamma B^T W^{-1} B)^{-1} B^T$ of $\mathcal{A}_\gamma$ is approximated by $-\frac{1}{\gamma} W$. In the related literature, several choices of W have been considered. In some studies (e.g., [3, 20]) W is chosen to be the identity matrix. In [8], a good choice for W is shown to be the pressure mass matrix M. Furthermore, it can be verified that it suffices to choose W as the diagonal of M, which is the practical choice in numerous applications and for many FE methods. For clarity, we briefly recall the analysis in [28]. Consider the following generalized eigenvalue problem

$$\mathcal{A}_\gamma \mathbf{v} = \lambda \mathcal{M}_L \mathbf{v}. \tag{3.5}$$

We see that

$$\mathcal{M}_L^{-1} \mathcal{A}_\gamma = \begin{bmatrix} I & (F + \gamma B^T W^{-1} B)^{-1} B^T \\ 0 & \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T \end{bmatrix}.$$

Thus, the eigenvalues λ in (3.5) are either equal to 1 (with multiplicity equal to the dimension of F) or coincide with those of the matrix $\gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T$. Applying Sherman-Morrison-Woodbury's formula to $(F + \gamma B^T W^{-1} B)^{-1}$, we have

$$\gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T = \gamma Q - \gamma Q (I + \gamma Q)^{-1} \gamma Q,$$

where $Q = W^{-1} B F^{-1} B^T$. The matrix $B F^{-1} B^T$ is the negative Schur complement of the original system matrix $\mathcal{A}$ in (2.4). Denote by $\mathcal{R}(\cdot)$ and $\mathcal{I}(\cdot)$ the real and the imaginary part of a complex number. We state the following theorem.

Theorem 3.1. *Let $\mu = a + i\,b$ be an eigenvalue of $Q = W^{-1} B F^{-1} B^T$ and λ be the eigenvalues of the eigenproblem (3.5). Let δ be an eigenvalue of the matrix $\widetilde{Q} = \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T$. Then the following holds:*

(1) *The matrices Q and $\widetilde{Q}$ have the same eigenvectors and the eigenvalues of $\widetilde{Q}$ are equal to*

$$\delta = \frac{\gamma \mu}{1 + \gamma \mu} = \frac{1}{1 + \frac{1}{\gamma \mu}}. \tag{3.6}$$

When $\gamma \to \infty$ all nonzero eigenvalues λ converge to 1.

(2) *Denote $\mu = a + i\,b$ and assume that μ is bounded. (Note that since $\mu \neq 0$ then a and b are not simultaneously equal to zero.) Then λ are also bounded and the following holds:*

$$\delta = 1 - \frac{1 + \gamma\, a}{(1 + \gamma\, a)^2 + \gamma^2 b^2} + i \frac{\gamma\, b}{(1 + \gamma\, a)^2 + \gamma^2 b^2}. \tag{3.7}$$

For any $\gamma \geq 1$, and any value of a and b, we have

$$1 - \frac{1 + \gamma |a|}{(1 + \gamma\, a)^2 + \gamma^2 b^2} < 1, \quad |\mathcal{I}(\delta)| = \frac{\gamma |b|}{(1 + \gamma\, a)^2 + \gamma^2 b^2} < 1. \tag{3.8}$$

If in addition $0 < a_1 \le a \le a_2$ *and* $|b| \le b_1$*, then*

$$\lambda_1 \le \mathcal{R}(\lambda) < 1 \quad \textit{and} \quad |\mathcal{I}(\lambda)| \le \lambda_2, \tag{3.9}$$

where $\lambda_1 = 1 - \dfrac{1}{1+\gamma a_1}$, $\lambda_2 = \dfrac{1}{2(1+\gamma a_1)^2}$.

(3) *Suppose that* a *and* b *are bounded as in* (2)*. Then, the asymptotic convergence factor* ϱ *of the* GMRES *method* [43] *when solving systems with* $\mathcal{A}_\gamma$ *in* (3.3)*, preconditioned by* $\mathcal{M}_L$*, is bounded as*

$$\begin{aligned} &\varrho \le \frac{\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}} = \frac{1}{\sqrt{1+4(\gamma a_1)^2(1+\gamma a_1)^2}}, && \textit{if } \frac{\lambda_1^2+\lambda_2^2}{\lambda_1} \ge \frac{1+\lambda_1}{2}, \\ &\varrho \le \sqrt{1+\frac{4\lambda_2^2-4\lambda_1}{(1+\lambda_1)^2}} = \frac{1}{(1+2\gamma a_1)}\sqrt{1+\frac{1}{(1+\gamma a_1)^2}}, && \textit{otherwise.} \end{aligned} \tag{3.10}$$

When $\gamma \to \infty$ *the asymptotic convergence factor* ϱ *converges to* 0*, which agrees with item* (2) *above.*

Proof. We prove that the above claims one by one

(1) The proof of this part can be found in [28], where it is used in the context of Oseen problem with constant viscosity.

(2) The expressions in (3.7) and (3.8) are straightforwardly observed. Further, we have

$$\begin{aligned} 1 > \mathcal{R}(\delta) &= \frac{\gamma a(1+\gamma a)+(\gamma b)^2}{(1+\gamma a)^2+(\gamma b)^2} \ge \frac{\gamma a(1+\gamma a)}{(1+\gamma a)^2} = \frac{\gamma a}{1+\gamma a} \ge \frac{\gamma a_1}{1+\gamma a_1}, \\ |\mathcal{I}(\delta)| &= \frac{\gamma|b|i}{(1+\gamma a)^2+(\gamma b)^2} \le \frac{\gamma|b|i}{(1+\gamma a_1)^2+(\gamma b)^2} = \frac{1}{\frac{(1+\gamma a_1)^2}{\gamma|b|}+\gamma|b|} \le \frac{1}{2(1+\gamma a_1)^2}, \end{aligned}$$

which shows the result in part (2).

(3) In the general case, the asymptotic convergence factor for GMRES iteration can be estimated based on an ellipse which contains all the eigenvalues of the coefficient matrix but not the origin (e.g., Theorem 4.2 in Chapter 4 in [23]). In our case, since the eigenvalues λ are contained in a single cluster, i.e., $0 < \lambda_1 \le \mathcal{R}(\lambda) \le 1$ and $|\mathcal{I}(\lambda)| \le \lambda_2$, instead of an ellipse, we consider a circle and use that to bound the asymptotic convergence factor, as is done in [15], for instance. The circle is positioned with a center $(x_0, 0)$ with $x_0 \ge (1+\lambda_1)/2$ and radius $r = \sqrt{(x_0-\lambda_1)^2+\lambda_2^2}$. To avoid containing the origin, we let $x_0 \ge r$, thus $x_0 \ge (\lambda_1^2+\lambda_2^2)/(2\lambda_1)$. Based on Theorem 5 in [43] the asymptotic convergence factor for GMRES is bounded as $\varrho \le r/x_0$. In this way, the task is transformed to the following minimization problem

$$\begin{aligned} \min y = \frac{r}{x_0} &= \sqrt{\frac{(x_0-\lambda_1)^2+\lambda_2^2}{x_0^2}} = \sqrt{(\lambda_1^2+\lambda_2^2)\frac{1}{x_0^2}-2\lambda_1\frac{1}{x_0}+1}, \\ &\text{such that } x_0 \ge \max\left\{\frac{1+\lambda_1}{2}, \frac{\lambda_1^2+\lambda_2^2}{2\lambda_1}\right\}. \end{aligned}$$

There exists a global minimum $y_{\min} = \lambda_2/\sqrt{\lambda_1^2+\lambda_2^2}$, attained for $x_0 = (\lambda_1^2+\lambda_2^2)/\lambda_1$. If $(\lambda_1^2+\lambda_2^2)/\lambda_1 \geq (1+\lambda_1)/2$, then

$$y_{\min} = \frac{\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}, \quad \text{when} \quad x_0 = \frac{\lambda_1^2+\lambda_2^2}{\lambda_1}.$$

If $(\lambda_1^2+\lambda_2^2)/\lambda_1 < (1+\lambda_1)/2$, then

$$y_{\min} = \sqrt{1+\frac{4\lambda_2^2-4\lambda_1}{(1+\lambda_1)^2}}, \quad \text{when} \quad x_0 = \frac{1+\lambda_1}{2}.$$

Supplying

$$\lambda_1 = \frac{\gamma a_1}{1+\gamma a_1} \quad \text{and} \quad \lambda_2 = \frac{1}{2(1+\gamma a_1)^2},$$

then item (3) is proved. □

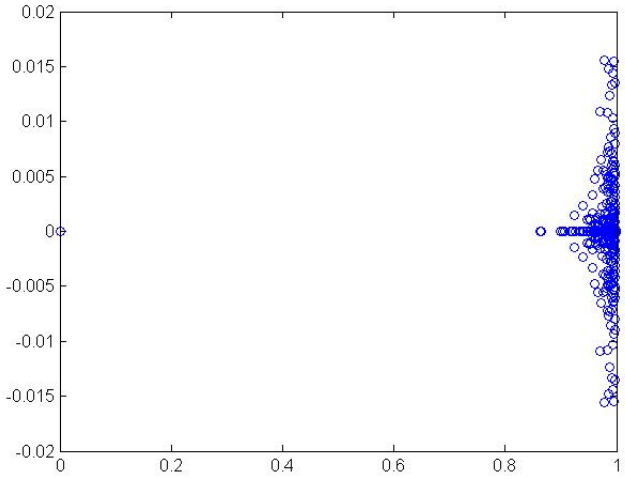
Theorem 3.1 explains the fact, observed in several other works, that $\gamma = 1$ suffices to ensure good numerical efficiency of the AL-preconditioner. It is also worth noticing that even if the bounds for μ may not be independent of h, the bounds for δ are. For Oseen problem, when ν decreases, a and $|b|$ get larger and they increase also when refining the mesh. This, however, improves the clustering of δ around 1, including the case when $\gamma = 1$, see Fig. 3.1(a). For large γ the clustering gets stronger, in particular $\mathcal{I}(\delta)$ decreases.When a and $|b|$ are large but γ is small, we see from the expression for $\mathcal{R}(\delta)$ that the ratio $(1+\gamma\, a)/((1+\gamma\, a)^2+\gamma^2 b^2)$ starts approaching 0, this, the clustering of the eigenvalues δ is shifted towards zero, see Fig. 3.1(b).

In this paper, for all numerical experiments we choose $\gamma = 1$ (or in some of the tests, slightly larger than one).

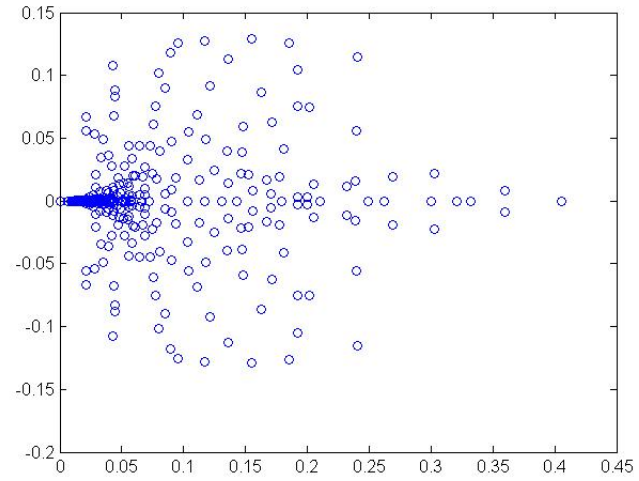
Remark 3.1. Theorem 3.1 provides an estimate of the asymptotic convergence rate of the preconditioned `GMRES` when solving systems with $\mathcal{A}_\gamma$ in (3.3) preconditioned by $\mathcal{M}_L$ in (3.4). The estimate in Theorem 3.1, part (3) has more theoretical value since it is well known that for nonsymmetric preconditioned iterations eigenvalues alone may not be sufficient to predict the convergence. A better tool could be to use the field of values technique, to more accurately predict the decrease of the relative residual, cf. [25], p. 56, for example. In [10], based on the field-of-values approach, for problems with constant viscosity and $W = M$, a theoretical analysis on the convergence rate of the AL-preconditioned GMRES is provided. For the case, when systems with the modified pivot block are solved exactly, the estimates show the order of γ related to ν^{-1}, which entails h-independent convergence, provided that bounds for μ are assumed to hold, as derived in [22]. It is also pointed out that, in all numerical tests, $\gamma = 1$ works sufficiently well.

Remark 3.2. For the case $W = M$, almost identical results as in Theorem 3.1 (1) and partly (2) are derived in [8], and in [9], Section 2.2. As we show here, the results hold for any positive definite matrix W. Result (1) in Theorem 3.1 can also be found in [28] and [13], for instance.

Next we show that the bounds for a and b, as required in part (2) of Theorem 3.1 hold true for the choices $W = M$ and $W = D_M$, where M is the finite element mass matrix for the pressure and D_M is its diagonal part. Further, it turns out that the estimate of the convergence factor, as shown in part (3) of Theorem 3.1, remains true for $W = D_M$ and $\gamma = 1$.



(a) $\gamma = 1, \nu_{\max} = 0.01, \nu_{\min} = 0.001$.



(b) $\gamma = 0.001, \nu_{\max} = 0.01, \nu_{\min} = 0.001$.

Fig. 3.1. Plot of the eigenvalues δ, $h = 1/64$.

Theorem 3.2. *For the choice $W = M$, the eigenvalues of the matrix $Q = M^{-1}BF^{-1}B^T$ are contained in a rectangular box in the right half complex plane, with boundaries independent of the mesh size h.*

Proof. Let $S = BF^{-1}B^T$, $C = B\left(\dfrac{F^{-1} + F^{-T}}{2}\right)B^T$ denote the symmetric part of S, and $R = B\left(\dfrac{F^{-1} - F^{-T}}{2}\right)B^T$ denote the skew-symmetric part of S. Thus, $S = C + R$. By Bendixson's theorem, the eigenvalues μ satisfy

$$\min_{\mathbf{p}} \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \leq \mathrm{Re}(\mu) \leq \max_{\mathbf{p}} \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \quad \text{and} \quad |\,\mathrm{Im}(\mu)\,| \leq \max_{\mathbf{p}} \frac{|\,(\mathbf{p}, R\mathbf{p})\,|}{(\mathbf{p}, M\mathbf{p})}.$$

We let $S' = BA_\nu^{-1}B^T$ and denote

$$\nu_{\min} = \inf_\Omega \nu(\mathbf{x}), \qquad \nu_{\max} = \sup_\Omega \nu(\mathbf{x}).$$

For the symmetric part C, for an arbitrary nonzero vector $\mathbf{p}$ we have

$$\frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} = \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} \frac{(\mathbf{p}, S'\mathbf{p})}{(\mathbf{p}, M\mathbf{p})}.$$

It has been proved in [26] that

$$c_0^2/\nu_{\max} \leq \frac{(\mathbf{p}, S'\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \leq 1/\nu_{\min}, \tag{3.11}$$

where the coefficient c_0 comes from the LBB condition. Next, we consider the symmetric part of F^{-1}:

$$\begin{aligned} \frac{F^{-1} + F^{-T}}{2} =& F^{-1} \frac{F + F^T}{2} F^{-T} \\ =& (A_\nu + N_s)^{-1} A_\nu (A_\nu - N_s)^{-1} \\ =& A_\nu^{-1/2} (I - \widetilde{N}^2)^{-1} A_\nu^{-1/2}, \end{aligned}$$

where $\widetilde{N} = A_\nu^{-1/2} N_s A_\nu^{-1/2}$. Thus

$$\frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} = \frac{(\mathbf{p}, BA_\nu^{-1/2}(I - \widetilde{N}^2)^{-1} A_\nu^{-1/2} B^T \mathbf{p})}{(\mathbf{p}, BA_\nu^{-1} B^T \mathbf{p})} = \frac{(\mathbf{q}, (I - \widetilde{N}^2)^{-1} \mathbf{q})}{(\mathbf{q}, \mathbf{q})},$$

where $\mathbf{q} = A_\nu^{-1/2} B^T \mathbf{p}$.

The matrix N_s is skew-symmetric, so is $\widetilde{N}$. Thus, the eigenvalues of $-\widetilde{N}^2$ are real and nonnegative. As mentioned in Section 2, see also the appendix, the modulus of the eigenvalues of $\widetilde{N}$ is bounded by $\frac{c_1}{\nu_{\min}}$, where c_1 is a constant independent of h. Therefore, the spectrum of $I - \widetilde{N}^2$ lies in the interval $[1, 1 + \frac{c_1^2}{\nu_{\min}^2}]$. Thus, we have

$$\frac{\nu_{\min}^2}{\nu_{\min}^2 + c_1^2} \leq \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} \leq 1.$$

The latter result, combined with (3.11), gives us the following bound

$$\frac{c_0^2 \nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2 + c_1^2)} \leq \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \leq \frac{1}{\nu_{\min}},$$

where the coefficients c_0, c_1, $\nu_{\min}$ and $\nu_{\max}$ do not depend on the mesh size h.

For the skew-symmetric part R, an analogous reasoning holds. First,

$$\frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} = \frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} \frac{(\mathbf{p}, S'\mathbf{p})}{(\mathbf{p}, M\mathbf{p})}.$$

Then, we have

$$\begin{aligned} \frac{F^{-1} - F^{-T}}{2} =& F^{-1} \left(\frac{F^T - F}{2} \right) F^{-T} \\ =& -(A_\nu + N_s)^{-1} N_s (A_\nu - N_s)^{-1} \\ =& -A_\nu^{-1/2} (I + \widetilde{N})^{-1} \widetilde{N}^{-1} (I - \widetilde{N})^{-1} A_\nu^{-1/2}, \end{aligned}$$

where $\widetilde{N} = A_\nu^{-1/2} N_s A_\nu^{-1/2}$. Therefore

$$\begin{aligned}\frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} &= -\frac{(\mathbf{p}, BA_\nu^{-1/2}(I+\widetilde{N})^{-1}\widetilde{N}(I-\widetilde{N})^{-1}A_\nu^{-1/2}B^T\mathbf{p})}{(\mathbf{p}, BA_\nu^{-1}B^T\mathbf{p})} \\ &= -\frac{(\mathbf{w}, \widetilde{N}\mathbf{w})}{(\mathbf{w}, (I-\widetilde{N}^2)\mathbf{w})},\end{aligned}$$

where $\mathbf{w} = (I-\widetilde{N})^{-1}A_\nu^{-1/2}B^T\mathbf{p}$. It is easy to prove that $\widetilde{N}$ is a normal matrix, so that it can be represented by a diagonal matrix Λ and unitary matrix U as

$$\widetilde{N} = iU\Lambda U^*,$$

where $\Lambda = \mathrm{diag}(-\sigma_1 i, -\sigma_2 i, \cdots)$ and σ is an eigenvalue of $\widetilde{N}$. Therefore, $\widetilde{N}^2 = -U\Lambda^2 U^*$, and the modulus of $-\dfrac{(\mathbf{w}, \widetilde{N}\mathbf{w})}{(\mathbf{w}, (I-\widetilde{N}^2)\mathbf{w})}$ can be written as

$$\left|\frac{(\mathbf{v}, \Lambda\mathbf{v})}{(\mathbf{v}, (I+\Lambda^2)\mathbf{v})}\right|,$$

where $\mathbf{v} = U^*\mathbf{w}$. For any $\mathbf{v}$, there exists a positive constant σ so that the following relation holds

$$\left|\frac{(\mathbf{v}, \Lambda\mathbf{v})}{(\mathbf{v}, (I+\Lambda^2)\mathbf{v})}\right| = \frac{\sigma}{1+\sigma^2} \quad (\sigma > 0).$$

Therefore, the modulus is bounded and the maximal upper bound is $1/2$, attained for $\sigma = 1$. Thus, we have

$$\left|\frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})}\right| \le \frac{1}{2}.$$

Again, the latter result, combined with (3.11), shows that

$$\frac{|\ (\mathbf{p}, R\mathbf{p})\ |}{(\mathbf{p}, M\mathbf{p})} \le \frac{1}{2\nu_{\min}}.$$

Thus, there holds

$$\frac{c_0^2\nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2 + c_1^2)} \le \mathrm{Re}(\mu) \le \frac{1}{\nu_{\min}} \quad \text{and} \quad |\ \mathrm{Im}(\mu)\ | \le \frac{1}{2\nu_{\min}}.$$

This completes the proof. □

Remark 3.3. The proof follows that of Theorem 1 in [22], where Oseen problem with constant viscosity is considered. Theorem 3.2 generalizes the result from Theorem 1 in [22], since for $\nu_{\min} = \nu_{\max} = \nu$, it is straightforward to obtain

$$\frac{c_0^2\nu}{(\nu^2 + c_1^2)} \le \mathrm{Re}(\mu) \le \frac{\Gamma}{\nu} \quad \text{and} \quad |\ \mathrm{Im}(\mu)\ | \le \frac{\Gamma}{2\nu}.$$

Of course, the values of the constants c_0, c_1 and Γ are different.

A broadly used practice is to replace M by its diagonal, D_M, which turns out to be a very good approximation of M. Choosing the matrix W to be D_M, the following proposition holds true.

Proposition 3.1. *For $W = D_M$, the eigenvalues of the matrix $Q = D_M^{-1}BF^{-1}B^T$ lie in a rectangular box in the right half plane, with boundaries independent of the mesh size h.*

Proof. Based on the results in [47], the following inequality holds true:

$$\alpha_1(\mathbf{p}, D_M\mathbf{p}) \leq (\mathbf{p}, M\mathbf{p}) \leq \alpha_2(\mathbf{p}, D_M\mathbf{p})$$

for any vector $\mathbf{p}$ of proper size. In 2D, for linear FEM on triangles, $\alpha_1 = 1/2, \alpha_2 = 2$ and on bilinear FEM rectangles $\alpha_1 = 1/4, \alpha_2 = 9/4$.

Following the proof of Theorem 3.2, it is straightforward to obtain

$$\frac{\alpha_1 c_0^2 \nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2 + c_1^2)} \leq \mathrm{Re}(\mu) \leq \frac{\alpha_2}{\nu_{\min}} \quad \text{and} \quad |\,\mathrm{Im}(\mu)\,| \leq \frac{\alpha_2}{2\nu_{\min}},$$

where μ are the eigenvalues of the matrix $Q = D_M^{-1}BF^{-1}B^T$. □

Remark 3.4. From this proposition we see that for $W = D_M$, part (2) in Theorem 3.1 still holds true. Furthermore, when iteratively solving the system in (3.3) using the AL-type preconditioner $\mathcal{M}_L$ in (3.4) with $W = D_M$, it is trivial to multiply a diagonal matrix with a vector, and the bounds of the asymptotic convergence factor for `GMRES` can be estimated based on part (3) in Theorem 3.1 and the convergence rate can be predicted. The corresponding bounds of the convergence factor are evaluated and given in the next section. Also we see that whether the variable viscosity is continuous or not does not impact the quality of the AL-type preconditioner $\mathcal{M}_L$ with $W = D_M$ and $W = M$, and that only the extremal values of the viscosity, i.e., $\nu_{\max}$ and $\nu_{\min}$ have an impact. This property makes the AL-type preconditioner applicable also in the case of discontinuous viscosity.

4. Numerical Illustrations

We choose as a benchmark the well-known two-dimensional lid-driven cavity problem, equipped with the boundary conditions $u_1 = u_2 = 0$ for $x = 0, x = 1$ and $y = 0$; $u_1 = 1, u_2 = 0$ for $y = 1$.

The problem is discretized using Cartesian meshes and the modified Taylor-Hood FE pair Q1isoQ1, i.e., piecewise bilinear basis functions on a mesh of size $2h$ for the pressure and piecewise bilinear basis functions for the velocity on a mesh of size h, obtained by one regular refinement of the discretization mesh for the pressure.

To linearize the stationary incompressible N-S equations we use Picard method, and the stopping tolerance for the nonlinear iterations in all experiments is relative and is chosen to be 10^{-6}. At each nonlinear step we solve the linear system (3.3), with the system matrix $\mathcal{A}_\gamma$ preconditioned by the AL-type preconditioner $\mathcal{M}_L$ from (3.4). For all experiments in this paper, $W = D_M$, where D_M is the diagonal part of the pressure mass matrix. We use `GMRES` as an iterative solution method to solve systems with $\mathcal{A}_\gamma$ and the corresponding iterations are referred to as the linear iterations. When solving systems with the preconditioner $\mathcal{M}_L$, for the modified pivot block matrix $F_\gamma = F + \gamma B^T W^{-1} B$, the results in Tables 4.2-4.4 are obtained using a direct solver and those in Tables 4.5-4.8 – using a preconditioned iterative method.

In this paper we consider two phase flow with variable viscosity and design two numerical experiments. Firstly, we divide the computational domain into two equal parts and, within each part, we set the viscosity to be constant, namely, $\nu_{left} = \nu_{\max}$ on the left part and $\nu_{right} = \nu_{\min}$

on the right part, where $\nu_{\max} \neq \nu_{\min}$. This setting models the stationary situation of two phases corresponding to a final stage of the phase separation process with an interface of zero width between them. We refer to this experiment as '*two phase flow at a final stage of the phase separation process*'. Compared to the final stage, in the beginning, the phases could be quite randomly spread in small islands in the domain. Therefore, in the second experiment, it is decided randomly whether the viscosity of each area in the domain is taken to be $\nu_{\min}$ or $\nu_{\max}$. Correspondingly, this experiment is referred to as '*two phase flow at an initial stage of the phase separation process*'. In these experiments, the values of the viscosity are set to be piecewise constant and discontinuous in the domain. As predicted by Theorem 3.1, Theorem 3.2 and Proposition 3.1, with the choice of $W = M$ or $W = D_M$, the convergence rate of the linear method depends on the extremal values of the viscosity, i.e., $\nu_{\max}$ and $\nu_{\min}$, but is independent of the continuity of the viscosity.

Table 4.1 shows the minimal nonzero and maximal eigenvalues of $D_M^{-1}S$ and the asymptotic convergence factor ϱ for two phase flow at a final stage of the phase separation process, where S is the negative Schur complement of the system matrix $\mathcal{A}$ from (2.4), i.e., $S = BF^{-1}B^T$. We observe that

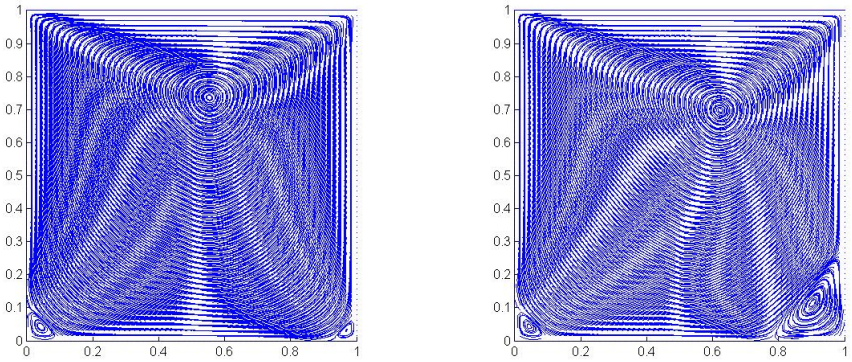
$$\mathrm{Re}(D_M^{-1}S)_{\min} = O(\nu_{\max}^{-1}), \quad \mathrm{Re}(D_M^{-1}S)_{\max} = O(\nu_{\min}^{-1}) \leq \frac{9}{4}\nu_{\min}^{-1},$$
$$|\ \mathrm{Im}(D_M^{-1}S)\ |_{\max} \leq \frac{9}{8}\nu_{\min}^{-1}.$$

Remark 4.1. The bounds for the $\mathrm{Re}(D_M^{-1}S)_{\max}$ and $|\ \mathrm{Im}(D_M^{-1}S)\ |_{\max}$ agree with the theoretical prediction given in Proposition 3.1. However, from Table 4.1, instead of the theoretical bound $\mathrm{Re}(D_M^{-1}S)_{\min} = O(\nu_{\max}^{-1}\frac{1}{1+c_1^2/\nu_{\min}^2})$, the relation $\mathrm{Re}(D_M^{-1}S)_{\min} = O(\nu_{\max}^{-1})$ is observed. A reason which could explain the observation is that the value of the constant c_1 turns out to be much smaller than the chosen values of $\nu_{\min}$ (0.05, 0.01, 0.005 and 10^{-3}), so that the ratio $\frac{1}{1+c_1^2/\nu_{\min}^2} \approx 1$ and $\mathrm{Re}(D_M^{-1}S)_{\min} = O(\nu_{\max}^{-1})$.

Table 4.2 and Table 4.3 show the nonlinear and linear iterations with varying mesh size for two phase flow at the final and initial stages of the phase separation process. Clearly, both the nonlinear and linear iterations are independent of the mesh size. In this paper, unless

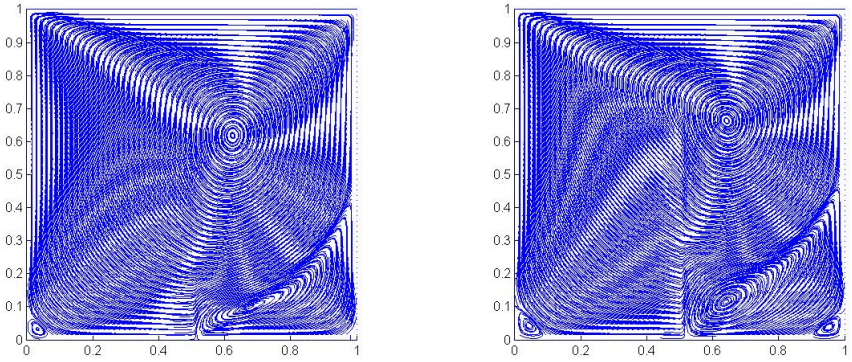
Table 4.1: Extreme eigenvalues of $D_M^{-1}S$ and asymptotic convergence factor for `GMRES` on the last nonlinear step, two phase flow at a final stage of the phase separation process, $\gamma = 1$.

	$\nu_{\max} = 0.1$			$\nu_{\max} = 0.01$	
$\nu_{\min}$	0.05	0.01	10^{-3}	0.005	10^{-3}
$h = 1/32$					
$\mathrm{Re}(D_M^{-1}S)_{\min}$	0.6350	0.6350	0.6350	6.3207	6.3210
$\mathrm{Re}(D_M^{-1}S)_{\max}$	18.9287	81.0884	766.0679	156.6134	671.669
$\lvert\ \mathrm{Im}(D_M^{-1}S)\ \rvert_{\max}$	1.6747	31.6947	191.905	67.0167	178.7441
convergence factor ϱ	0.5164	0.5164	0.5164	0.0734	0.0740
$h = 1/64$					
$\mathrm{Re}(D_M^{-1}S)_{\min}$	0.6350	0.6349	0.6349	6.3436	6.3436
$\mathrm{Re}(D_M^{-1}S)_{\max}$	20.8379	97.4026	935.7287	191.6925	901.2033
$\lvert\ \mathrm{Im}(D_M^{-1}S)\ \rvert_{\max}$	2.3942	40.9770	287.5361	86.2511	294.5081
convergence factor ϱ	0.5164	0.5164	0.5164	0.0737	0.0737



(a) $\nu_{\max} = 1$, $\nu_{\min} = 0.1$.

(b) $\nu_{\max} = 1$, $\nu_{\min} = 10^{-2}$.



(c) $\nu_{\max} = 0.1$, $\nu_{\min} = 10^{-3}$.

(d) $\nu_{\max} = 1$, $\nu_{\min} = 10^{-3}$.

Fig. 4.1. Two phase flow at a final stage of the phase separation process: selected streamlines, $h = \frac{1}{64}$.

explicitly stated, the stopping tolerance for the linear iterations is relative and is chosen to be 10^{-2}. Tighter values are not necessary because the convergence of the nonlinear iterations is not improved by decreasing the (relative) stopping tolerance for the linear systems (see also the numerical experiments in [29]). Thus, if one solves with the modified pivot block exactly, or accurately enough, a relative stopping tolerance of 10^{-2} for the linear solutions suffices.

The asymptotic convergence factor ϱ for two phase flow at a final stage of the phase separation process, shown in Table 4.1, is estimated based on the bounds (3) in Theorem 3.1. The convergence of the whole method is illustrated in Table 4.2. Fig. 4.2 shows the `GMRES` convergence history of the linear iterations on the last nonlinear step. Theorem 3.1 and Proposition 3.1 predict that, with $W = D_M$, the factor ϱ depends on $\nu_{\min}$ and $\nu_{\max}$. However, the extremal eigenvalues presented in Table 4.1 show that the factor ϱ depends on $\nu_{\max}$, instead of $\nu_{\min}$, in the chosen range of $\nu_{\max}$ and $\nu_{\min}$. In other words, smaller values of $\nu_{\max}$ result in smaller values of ϱ, corresponding to a faster linear convergence rate.

Fig. 4.1 shows some selected streamlines of the velocity on the last nonlinear iteration for two phase flow at a final stage of the phase separation process with different values of $\nu_{\min}$ and $\nu_{\max}$.

Simplifying the task to approximate the Schur complement of the original system, the AL

Table 4.2: Two phase flow at a final stage of the phase separation process: iterations to solve $\mathcal{A}_\gamma$ preconditioned by $\mathcal{M}_L$, $\gamma = 1$.

	$\nu_{\max} = 0.1$			$\nu_{\max} = 0.01$	
$\nu_{\min}$	0.05	0.01	10^{-3}	0.005	10^{-3}
$h = 1/32$					
nonlinear iter.	5	7	17	12	21
linear iter.	5	5	5	3	3
$h = 1/64$					
nonlinear iter.	5	7	13	12	21
linear iter.	5	5	5	3	3
$h = 1/128$					
nonlinear iter.	4	6	12	11	19
linear iter.	5	5	5	3	3
$h = 1/256$					
nonlinear iter.	4	6	12	11	19
linear iter.	5	5	5	3	3

technique shifts the difficulty to solving systems with the modified pivot block $F_\gamma = F + \gamma B^T W^{-1} B$, which itself is a Schur complement. The analysis in Theorem 3.1 and significant numerical evidence (e.g., [8–10, 13, 28, 30]) show that first, in order to achieve the efficiency of the (outer) block-triangular preconditioner $\mathcal{M}_L$, systems with F_γ have to be solved accurately enough and second, that $\gamma = 1$ is the preferred choice of the stabilization parameter. Direct solution methods for F_γ are clearly unfeasible for large scale computations. While striving to find an efficient preconditioner for F_γ, in order to enable its solution via iterative methods, many

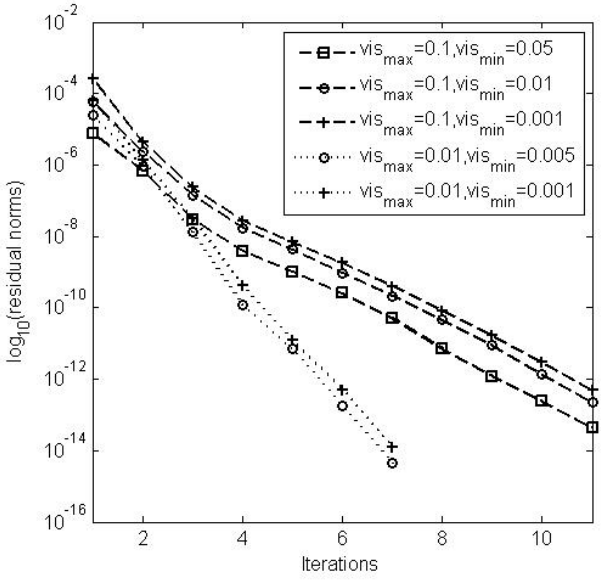


Fig. 4.2. Two phase flow at a final stage of the phase separation process: convergence history of `GMRES` on the last nonlinear step, $\gamma = 1$, $h = \frac{1}{32}$.

Table 4.3: Two phase flow at an initial stage of the phase separation process: iterations to solve $\mathcal{A}_\gamma$ preconditioned by $\mathcal{M}_L$, $\gamma = 1$.

	$\nu_{\max} = 0.1$			$\nu_{\max} = 0.01$	
$\nu_{\min}$	0.05	0.01	10^{-3}	0.005	10^{-3}
$h = 1/32$					
nonlinear iter.	5	6	7	11	16
linear iter.	5	5	5	3	3
$h = 1/64$					
nonlinear iter.	5	6	7	11	16
linear iter.	5	5	5	3	3
$h = 1/128$					
nonlinear iter.	5	6	7	11	16
linear iter.	5	5	5	3	3
$h = 1/256$					
nonlinear iter.	5	6	7	11	16
linear iter.	5	5	5	3	3

Table 4.4: Iterations and time (sec) to solve $\mathcal{A}_\gamma$ preconditioned by $\mathcal{M}_L$ with $F_\gamma/\widetilde{F}_\gamma$: $\gamma = 10$, systems with F_γ and $\widetilde{F}_\gamma$ solved directly, time (sec).

$\nu_{\max}$	0.1			0.01	
$\nu_{\min}$	0.05	0.01	0.001	0.005	0.001
$h = 1/64$					
nonlin. iter.	5/5	7/6	13/12	12/11	21/20
linear iter.	2/3	1/3	1/2	1/3	1/3
nonlin.solver	9.5/13.0	11.9/16.1	22.6/29.6	20.4/26.4	35.6/58.0
$h = 1/128$					
nonlin. iter.	4/4	6/6	12/11	11/10	19/18
linear iter.	2/3	1/3	1/2	1/2	1/3
nonlin.solver	48.8/69.0	70.4/109.1	143.3/185.5	132.7/161.9	231.1/350.5

ideas have been developed, in which however, γ is suggested to be chosen rather small, sometimes of order 0.001 (e.g., [9, 13]). The latter, in its turn, means that we try to precondition F_γ by F, which in general cannot be efficient and entails increase in the number of outer iterations.

Without considering this problem to be fully resolved, we include experiments with an algebraic multigrid (AMG) method and $\gamma = 1$ and also slightly larger than 1 to illustrate that it is possible to solve F_γ efficiently without slowing down the numerical performance of the outer solver. The AMG method we use is `agmg`, see [35, 38, 39]. The implementation is in `Fortran` and `Matlab` interface is provided. Therefore, its performance in time is comparable with that of the 'backslash' direct solver in `Matlab`. The `agmg` solver is used here as a black box. For problems with constant, but small viscosity, it exhibits extraordinary good performance. It converges in one iteration and acts as a direct solver. Straightforwardly applied to problems with variable viscosity, it is found somewhat less efficient.

We note that, in contrast to the constant viscosity case, the block F is not block-diagonal anymore. It can be seen as a sum of two parts, $F = F_D + F_V$, where F_D is the block-diagonal

Table 4.5: Solving systems with $F_\gamma/\widetilde{F}_\gamma$ by `agmg`: $\gamma = 10$, relative stopping tolerance 10^{-6}, random right hand vector, time (sec).

$\nu_{\max}$	0.1			0.01	
$\nu_{\min}$	0.05	0.01	0.001	0.005	0.001
$size(F_\gamma) = 8450$					
`agmg` iterations	18/1	20/1	23/1	1/1	1/1
`agmg` setup time	0.44/0.44	0.44/0.49	0.45/0.45	0.45/0.45	0.45/0.45
`agmg` solution time	0.31/0.02	0.36/0.02	0.40/0.02	0.02/0.02	0.02/0.02
`agmg` total time	0.75/0.46	0.80/0.51	0.85/0.47	0.47/0.47	0.47/0.47
Direct solver	0.51/0.53	0.51/0.57	0.54/0.53	0.52/0.53	0.51/0.51
$size(F_\gamma) = 33282$					
`agmg` iterations	47/1	57/1	69/1	1/1	1/1
`agmg` setup time	2.55/2.78	2.58/2.77	2.59/2.82	2.71/2.74	2.80/2.87
`agmg` solution time	4.78/0.11	5.71/0.11	6.82/0.11	0.11/0.11	0.14/0.14
`agmg` total time	7.33/2.89	8.29/2.88	9.41/2.93	2.82/2.85	2.94/3.01
Direct solver	3.21/3.19	3.20/3.24	3.28/3.32	3.22/3.08	3.41/3.37

part and $[F_D]_{i,j} = (\nu(\mathbf{x})\nabla\vec{\varphi}_i, \nabla\vec{\varphi}_j)$. Correspondingly, we denote $\widetilde{F}_\gamma = F_D + \gamma B^T W^{-1} B$. We illustrate that `agmg` is a viable option for the considered class of problems.

The results in Tables 4.4-4.8 are all computed for the two phase flow at a final stage of the phase separation process. First we show that the outer solution method preserves its efficiency when $\mathcal{M}_L$ is constructed with either F_γ or $\widetilde{F}_\gamma$ (Table 4.4). Next, in Table 4.7, we show the behaviour of `agmg` when solving $\widetilde{F}_\gamma$ and point out that $\gamma > 1$ ($\gamma = 10$) is better than $\gamma = 1$. In Table 4.5 we compare the solution time for F_γ and $\widetilde{F}_\gamma$ using `agmg` and the direct method, and see that `agmg`, applied to $\widetilde{F}_\gamma$ becomes slightly faster than the direct method, already for relatively small problems.

Table 4.6 shows a gain in overall time for the nonlinear solver. The important detail is that the `agmg` preconditioner is constructed once every 3 nonlinear iterations (and when $\nu_{\max} = 0.01$, $\nu_{\min} = 0.001$ - every 5 nonlinear steps).

Finally, Table 4.8 indicates the expected savings in overall solution time when the problem size is increased, where $\textit{Gain in time} = \frac{T_{direct} - T_{agmg}}{T_{direct}}$.

Table 4.6: Iterations and time (sec) to solve $\mathcal{A}_\gamma$ preconditioned by $\mathcal{M}_L$ with $F_\gamma/\widetilde{F}_\gamma$: $\gamma = 10$, systems with F_γ solved directly and with $\widetilde{F}_\gamma$ solved by `agmg`, time (sec).

$\nu_{\max}$	0.1			0.01	
$\nu_{\min}$	0.05	0.01	0.001	0.005	0.001
$h = 1/64$					
nonlinear iter.	5/5	7/6	13/12	12/11	21/20
linear iter.	2/3	1/3	1/2	1/3	1/3
nonlin. solver time	9.5/3.0	11.9/3.8	22.6/6.0	20.4/5.2	35.6/10.8
$h = 1/128$					
nonlinear iter.	4/4	6/6	12/11	11/10	19/18
linear iter.	2/3	1/3	1/2	1/2	1/3
nonlin. solver time	48.8/23.6	70.4/32.4	143.3/56.0	132.7/46.7	231.1/99.1

Table 4.7: Solving systems with $\widetilde{F}_\gamma$ by `agmg` for $\gamma = 1/\gamma = 10$: relative stopping tolerance 10^{-6}, random right hand vector, time (sec).

$\nu_{\max}$	0.1			0.01	
$\nu_{\min}$	0.05	0.01	0.001	0.005	0.001
$size(F_\gamma) = 8450$					
`agmg` iterations	23/1	22/1	25/1	1/1	1/1
`agmg` setup time	0.22/0.44	0.40/0.49	0.44/0.45	0.48/0.45	0.48/0.45
`agmg` solution time	0.38/0.02	0.38/0.02	0.43/0.02	0.03/0.02	0.03/0.02
`agmg` total time	0.60/0.46	0.78/0.51	0.87/0.47	0.51/0.47	0.51/0.47
$size(F_\gamma) = 33282$					
`agmg` iterations	60/1	59/1	61/1	1/1	1/1
`agmg` setup time	1.62/2.78	2.13/2.77	2.09/2.82	2.78/2.74	2.80/2.87
`agmg` solution time	4.14/0.11	5.26/0.11	5.28/0.11	0.13/0.11	0.13/0.14
`agmg` total time	5.76/2.89	7.39/2.88	7.37/2.93	2.91/2.85	2.93/3.01

Remark 4.2. The efficiency of the `agmg` method, originally developed for scalar problems and nonsymmetric M-matrices, could be related to the fact that `agmg` assumes 'divergence-free' solutions, which is the case when solving incompressible Navier-Stokes equations.

Table 4.8: Solving $\widetilde{F}_\gamma$ directly and by `agmg`: $\gamma = 10$, $\nu_{\max} = 0.01$, $\nu_{\min} = 0.001$, relative stopping tolerance (`agmg`) 10^{-12}, random right hand vector, time (sec).

$size(\widetilde{F}_\gamma)$	8450	33282	132098	526338
`agmg` iterations	1	1	1	1
`agmg` total time	0.47	3.01	16.70	127.70
Direct solver time	0.51	3.37	20.68	187.99
Gain in time	7%	10%	20%	32%

5. Discussion and Conclusions

The AL approach, used in this paper and also in other works, such as [8,9,28], is also known as the '*First-Discretize-Then-Stabilize*' (FDTS) technique. There are also other techniques, known as '*First-Stabilize-Then-Discretize*' (FSTD). To illustrate those, consider the stationary Oseen problem with constant viscosity

$$-\nu\nabla\cdot(\nabla\mathbf{u}) + (\mathbf{w}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{5.1a}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{5.1b}$$

If we apply the gradient operator to the divergence-free constraint in (5.1), and add the so-obtained result, pre-multiplied with a stabilization constant γ to the first equation we obtain the so-called '*grad-div*' stabilization formulation (see, e.g., [13, 30])

$$-\nu\nabla\cdot(\nabla\mathbf{u}) + (\mathbf{w}\cdot\nabla)\mathbf{u} - \gamma\nabla(\nabla\cdot\mathbf{u}) + \nabla p = \mathbf{f}, \tag{5.2a}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{5.2b}$$

For completeness, we mention that, alternatively, one can apply the divergence operator to the first equation in (5.1), which leads to the so-called pressure equation

$$\nabla \cdot (\nabla p) = \nabla \cdot \mathbf{f},$$

which, pre-multiplied by a stabilization constant γ, is then added to the second equation in (5.1) and the system becomes

$$-\nu \nabla \cdot (\nabla \mathbf{u}) + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{5.3a}$$

$$\nabla \cdot \mathbf{u} + \gamma \nabla \cdot (\nabla p) = \gamma \nabla \cdot \mathbf{f}. \tag{5.3b}$$

A pressure correction stabilization for the Stokes equation has been used, for instance, in [2], to enable the usage of a unstable FEM discretization.

The difference between FSTD and FDTS stabilization, applied to the first equation in (5.1) is in the pivot block of the resulting matrix, namely $F_\gamma^{FSTD} = F + \gamma \widehat{F}$ and $F_\gamma^{FDTS} = F + \gamma B^T W^{-1} B$. It is clear (also noted in [30]) that F_γ^{FSTD} is sparser than F_γ^{FDTS}. On the other side, $\widehat{F}$ denotes the discrete operator of $(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$ ($\mathbf{v}$ is the test function), and is analogous to the matrix $B^T B$. Thus, the matrix F_γ^{FSTD} is analogous to F_γ^{FDTS} with W - the identity matrix. The latter indicates that the means to tune the FSTD stabilization is only the constant γ, while in FDTS we possess γ and W to play with. So far, the most natural and easily available choice of W has been the pressure mass matrix M, or, rather, its diagonal. This, however, does not fully discard the need to look for other choices of W, which approximate $BF^{-1}B^T$ better than M, especially when the exact Schur complement $BF^{-1}B^T$ is nonsymmetric.

In this paper we consider the augmented Lagrangian approach to precondition matrices arising from the finite element discretization of the linearized stationary incompressible Navier-Stokes equations with variable viscosity. We prove that the AL preconditioner involving the finite element mass matrix for the pressure is fully robust with respect to the mesh size, but depends on the maximal and minimal values of the viscosity, i.e., $\nu_{\max}$ and $\nu_{\min}$. The presented results are more general and the estimates in [22] can be obtained as a special case by taking $\nu_{\max} = \nu_{\min} = \nu$.

Further, provided that systems with the modified pivot block are solved accurately enough, we also show that any $\gamma \geq 1$ suffices to ensure high numerical efficiency of the AL-preconditioner.

The numerical experiments in Section 4 illustrate that the AL preconditioner works efficiently for a broad range of values of $\nu_{\max}$ and $\nu_{\min}$. The observation in Remark 4.1, namely, that the conducted numerical tests do not show any dependence of $\mathrm{Re}(D_M^{-1} S)$ on $\nu_{\min}$, needs further consideration.

An issue of particular importance is how to efficiently solve the modified pivot block of the transformed system matrix. Here we show that a general purpose AMG solver, namely `agmg`, used as a black-box, can be successfully applied for the full range of the problem and discretization parameters, considered in this paper.

Appendix

Consider the form $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \frac{1}{2}(c(\mathbf{w}, \mathbf{u}, \mathbf{v}) - c(\mathbf{w}, \mathbf{v}, \mathbf{u}))$, where $c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v})$, and the operator $l(\mathbf{u}, \mathbf{v}) = (2\nu(\mathbf{x})\mathbf{D}\mathbf{u}, \mathbf{D}\mathbf{v})$, where $\mathbf{D}\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$. The following results holds.

Proposition A.1. *Consider the above operators l and $\tilde{c}$. Assume that $\boldsymbol{w}$ and $\nu(\boldsymbol{x})$ are bounded in Ω and that the part of $\partial\Omega$, where Dirichlet conditions for velocity $\boldsymbol{u}$ are imposed, is not empty. Then there holds,*

$$\| l^{-1}\tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \|_{L_2} \leq \frac{C}{\sqrt{\nu_{\min}}}. \tag{A.1}$$

Proof. We have

$$\begin{aligned} \| l(\mathbf{u},\mathbf{u}) \|_{L_2}^2 &= \int_\Omega 2\nu \mid \mathbf{Du} \mid^2 \\ &\geq 2\nu_{\min} \int_\Omega \mid \mathbf{Du} \mid^2 \geq C_1 \nu_{\min} \| \nabla \mathbf{u} \|_{L_2}^2 . \end{aligned} \tag{A.2}$$

For the last part of relation (A.2) we use Korn's inequality. Further,

$$\begin{aligned} \| \tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \|_{L_2}^2 &= \int_\Omega \mid \tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \mid \\ &\leq C_2 \| \mathbf{w} \|_\infty \| \nabla \mathbf{u} \|_{L_2} \| \mathbf{u} \|_{L_2} \\ &\leq C_3 \| \mathbf{w} \|_\infty \| \nabla \mathbf{u} \|_{L_2}^2 . \end{aligned} \tag{A.3}$$

For the last part of relation (A.3) we use the assumption that $\partial\Omega_D \neq \emptyset$. Combing (A.2) and (A.3), we obtain,

$$\| \tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \|_{L_2}^2 \leq \frac{C}{\nu_{\min}} \| l(\mathbf{u},\mathbf{u}) \|_{L_2}^2, \tag{A.4}$$

which yields (A.1). □

Acknowledgments. The work of the second author is (partly) supported by the Swedish Research Council (VR) via the grant *Finite element preconditioners for algebraic problems as arising in modelling of multiphase microstructures*, 2009-2011. The support is hereby gratefully acknowledged. The authors thank the anonymous reviewers for their constructive criticism, which helped to improve the presentation, and to refer to related research. Special acknowledgement goes to Yvan Notay for making the `agmg` solver publicly available. Most of the computations were performed on resources provided by SNIC through Uppsala Multidisciplinary Center for Advanced Computational Science (UPPMAX) under Project p2009040.

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