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BANDED TOEPLITZ PRECONDITIONERS FOR TOEPLITZ MATRICES FROM SINC METHODS *

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Abstract

We give general expressions, analyze algebraic properties and derive eigenvalue bounds for a sequence of Toeplitz matrices associated with the sinc discretizations of various orders of differential operators. We demonstrate that these Toeplitz matrices can be satisfactorily preconditioned by certain banded Toeplitz matrices through showing that the spectra of the preconditioned matrices are uniformly bounded. In particular, we also derive eigenvalue bounds for the banded Toeplitz preconditioners. These results are elementary in constructing high-quality structured preconditioners for the systems of linear equations arising from the sinc discretizations of ordinary and partial differential equations, and are useful in analyzing algebraic properties and deriving eigenvalue bounds for the corresponding preconditioned matrices. Numerical examples are given to show effectiveness of the banded Toeplitz preconditioners.

Mathematics subject classification: 65F10, 65F50.
Key words: Toeplitz matrix, Banded Toeplitz preconditioner, Generating function, Sinc method, Eigenvalue bounds.

1. Introduction

Let $\mathbb{L}^2[-\pi,\pi]$ be the functional space of all quadratically integrable functions defined on the interval $[-\pi,\pi]$. For $f \in \mathbb{L}^2[-\pi,\pi]$, denote by

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\imath k\theta}\mathrm{d}\theta, \quad k \in \mathbb{Z},$$

the Fourier coefficients of f, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ represents the set of all integers and $\imath$ denotes the imaginary unit. For all $n \geq 1$, we write $A_n = (a_{j,k})$ the n-by-n Toeplitz matrix with entries satisfying $a_{j,k} = a_{j-k}$, $1 \leq j,k \leq n$. The function f is called the generating function of the sequence of Toeplitz matrices A_n, $n = 1, 2, \ldots$. Alternatively, we also use $A_n[f]$ to denote the n-by-n Toeplitz matrix generated by the function f. Note that $A_n[f]$ is a non-Hermitian matrix when f is a complex-valued function, and it is a Hermitian matrix when f is a real-valued function. In particular, if f is real-valued and even, then $A_n[f]$ is real symmetric.

Toeplitz systems of linear equations arise in a variety of applications in mathematics and engineering. In particular, when the sinc method is applied to discretize the linear ordinary and partial differential equations, we can often obtain systems of linear equations whose coefficient matrices are combinations of Toeplitz and diagonal matrices; see [1–3, 10, 14]. Hence, it

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is a basic requirement to discuss the algebraic properties of these Toeplitz matrices, construct their effective preconditioners, and derive tight eigenvalue bounds for the corresponding preconditioned matrices. Here, we construct banded Toeplitz preconditioners by making use of trigonometric generating functions, which is an idea first proposed in [6] for Toeplitz matrices with nonnegative generating functions.

In this paper, we first give the general expressions for the Toeplitz matrices associated with the sinc discretizations of various orders of differential operators, and derive the generating functions of these Toeplitz matrices as well as their eigenvalue bounds. According to suitable approximations to the generating functions, we construct banded Toeplitz preconditioners and demonstrate the uniformly bounded property about the spectra of the preconditioned matrices. In particular, we also derive eigenvalue bounds for the banded Toeplitz preconditioners. These results are elementary in constructing high-quality structured preconditioners for the systems of linear equations arising from the sinc discretizations of ordinary and partial differential equations, and are useful in analyzing algebraic properties and deriving eigenvalue bounds for the corresponding preconditioned matrices.

The outline of the paper is as follows. In Section 2, we derive the expression of the Toeplitz matrices from sinc methods and their properties. In Section 3, we construct the banded Toeplitz preconditioners and analyze the eigenvalue bounds for these preconditioners. Some useful bounds are established for the Toeplitz matrices and the banded preconditioners in Section 4. In Section 5, numerical examples are given to show the effectiveness of the banded Toeplitz preconditioners.

2. Toeplitz Matrices

The sinc function is defined as

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \quad -\infty < t < \infty,$$

and the corresponding sinc basis functions are given by

$$S(j,h)(t) := \frac{\sin[\pi(t - jh)/h]}{\pi(t - jh)/h}, \quad -\infty < t < \infty, \quad j \in \mathbb{Z},$$

where h is the step-size used in sinc methods [14]. The points $t_j = jh$ $(j \in \mathbb{Z})$ are called the sinc-grid points. In sinc methods, we also need to introduce a one-to-one conformal mapping, say, $\phi(x)$, which maps a simply-connected domain onto a strip region.

The n-by-n Toeplitz matrices associated with sinc discretizations of the linear ordinary and partial differential equations are of the form

$$T^{(m)} \equiv \big[\delta_{jk}^{(m)}\big], \qquad j,k \in \mathbb{Z}, \quad m = 0,1,2,\ldots, \tag{2.1}$$

where $\delta_{jk}^{(m)}$ is defined as

$$\delta_{jk}^{(m)} := h^m \frac{\mathrm{d}^m}{\mathrm{d}\phi^m}[S(j,h) \circ \phi(x)]\bigg|_{x=x_k}. \tag{2.2}$$

For example, $T^{(0)} = I$ is the identity matrix, and for $m = 1,2,3,4$ the Toeplitz matrices $T^{(m)}$

have the actual expressions

$$T^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \dots & \frac{(-1)^{n-1}}{n-1} \\ 1 & 0 & \ddots & \ddots & \vdots \\ -\frac{1}{2} & 1 & \ddots & -1 & \frac{1}{2} \\ \vdots & \ddots & \ddots & 0 & -1 \\ -\frac{(-1)^{n-1}}{n-1} & \dots & -\frac{1}{2} & 1 & 0 \end{bmatrix}, \tag{2.3}$$

$$T^{(2)} = \begin{bmatrix} -\frac{\pi^2}{3} & 2 & -\frac{2}{2^2} & \dots & \frac{2(-1)^n}{(n-1)^2} \\ 2 & -\frac{\pi^2}{3} & \ddots & \ddots & \vdots \\ -\frac{2}{2^2} & 2 & \ddots & 2 & -\frac{2}{2^2} \\ \vdots & \ddots & \ddots & -\frac{\pi^2}{3} & 2 \\ \frac{2(-1)^n}{(n-1)^2} & \dots & -\frac{2}{2^2} & 2 & -\frac{\pi^2}{3} \end{bmatrix}, \tag{2.4}$$

$$T^{(3)} = \begin{bmatrix} 0 & -(6-\pi^2) & \frac{6-2^2\pi^2}{2^3} & \dots & \frac{(-1)^{n-1}[6-(n-1)^2\pi^2]}{(n-1)^3} \\ 6-\pi^2 & 0 & \ddots & \ddots & \vdots \\ \frac{-(6-2^2\pi^2)}{2^3} & 6-\pi^2 & \ddots & -(6-\pi^2) & \frac{6-2^2\pi^2}{2^3} \\ \vdots & \ddots & \ddots & 0 & -(6-\pi^2) \\ \frac{(-1)^n[6-(n-1)^2\pi^2]}{(n-1)^3} & \dots & \frac{-(6-2^2\pi^2)}{2^3} & 6-\pi^2 & 0 \end{bmatrix}, \tag{2.5}$$

and

$$T^{(4)} = \begin{bmatrix} \frac{\pi^4}{5} & 4(6-\pi^2) & \frac{-4(6-2^2\pi^2)}{2^4} & \dots & \frac{4(-1)^n[6-(n-1)^2\pi^2]}{(n-1)^4} \\ 4(6-\pi^2) & \frac{\pi^4}{5} & \ddots & \ddots & \vdots \\ \frac{-4(6-2^2\pi^2)}{2^4} & 4(6-\pi^2) & \ddots & 4(6-\pi^2) & \frac{-4(6-2^2\pi^2)}{2^4} \\ \vdots & \ddots & \ddots & \frac{\pi^4}{5} & 4(6-\pi^2) \\ \frac{4(-1)^n[6-(n-1)^2\pi^2]}{(n-1)^4} & \dots & \frac{-4(6-2^2\pi^2)}{2^4} & 4(6-\pi^2) & \frac{\pi^4}{5} \end{bmatrix}.$$

The following lemma presents intuitive expressions for the elements $\delta_{jk}^{(m)}$ of the Toeplitz matrices $T^{(m)}$, $m = 0, 1, 2, \dots$.

Lemma 2.1. *Let ϕ be an one-to-one conformal mapping from a simply connected domain onto a strip region, and $\delta_{jk}^{(m)} (m = 0, 1, 2, \dots)$ be defined as in (2.2), with $x_k = \phi^{-1}(kh)$. Then, for $m = 0, 1, 2, \dots$, it holds that*

$$\delta_{jk}^{(4m)} = \begin{cases} \frac{\pi^{4m}}{4m+1}, & j = k, \\ \left(\frac{4m\pi^{(4m-2)}}{(k-j)^2} - \frac{4m(4m-1)(4m-2)\pi^{(4m-4)}}{(k-j)^4} + \dots - \frac{4m(4m-1)\cdots 2}{(k-j)^{4m}} \right) (-1)^{k-j}, & j \neq k; \end{cases}$$

$$\delta_{jk}^{(4m+1)}=\begin{cases}0, \quad j=k,\\ \left(\frac{\pi^{4m}}{k-j}-\frac{(4m+1)4m\pi^{(4m-2)}}{(k-j)^3}+\cdots+\frac{(4m+1)4m\cdots 2}{(k-j)^{4m+1}}\right)(-1)^{k-j}, \quad j\neq k;\end{cases}$$

$$\delta_{jk}^{(4m+2)}=\begin{cases}-\frac{\pi^{(4m+2)}}{4m+3}, \quad j=k,\\ \left(-\frac{(4m+2)\pi^{4m}}{(k-j)^2}+\frac{(4m+2)(4m+1)4m\pi^{(4m-2)}}{(k-j)^4}-\cdots-\frac{(4m+2)(4m+1)\cdots 2}{(k-j)^{4m+2}}\right)(-1)^{k-j}, \quad j\neq k;\end{cases} \tag{2.6}$$

$$\delta_{jk}^{(4m+3)}=\begin{cases}0, \quad j=k,\\ \left(-\frac{\pi^{(4m+2)}}{k-j}+\frac{(4m+3)(4m+2)\pi^{4m}}{(k-j)^3}-\cdots+\frac{(4m+3)(4m+2)\cdots 2}{(k-j)^{4m+3}}\right)(-1)^{k-j}, \quad j\neq k.\end{cases}$$

Proof. Because

$$\frac{\mathrm{d}^m}{\mathrm{d}\phi^m}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k}=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\mathrm{sinc}\left(\frac{x-jh}{h}\right)\right]\Big|_{x=kh} \tag{2.7}$$

and

$$\mathrm{sinc}\left(\frac{x-jh}{h}\right)=\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}e^{-\imath xt}e^{\imath jht}\,\mathrm{d}t, \tag{2.8}$$

by combining (2.2), (2.7) and (2.8) we can obtain the formulas

$$\begin{aligned}
\frac{\mathrm{d}^{4m}}{\mathrm{d}x^{4m}}\left[\mathrm{sinc}\left(\frac{x-jh}{h}\right)\right]&=\frac{h}{\pi}\int_{-\frac{\pi}{h}}^{0}t^{4m}\cos[t(x-jh)]\,\mathrm{d}t,\\
\frac{\mathrm{d}^{4m+1}}{\mathrm{d}x^{4m+1}}\left[\mathrm{sinc}\left(\frac{x-jh}{h}\right)\right]&=\frac{-h}{\pi}\int_{-\frac{\pi}{h}}^{0}t^{4m+1}\sin[t(x-jh)]\,\mathrm{d}t,\\
\frac{\mathrm{d}^{4m+2}}{\mathrm{d}x^{4m+2}}\left[\mathrm{sinc}\left(\frac{x-jh}{h}\right)\right]&=\frac{-h}{\pi}\int_{-\frac{\pi}{h}}^{0}t^{4m+2}\cos[t(x-jh)]\,\mathrm{d}t,\\
\frac{\mathrm{d}^{4m+3}}{\mathrm{d}x^{4m+3}}\left[\mathrm{sinc}\left(\frac{x-jh}{h}\right)\right]&=\frac{h}{\pi}\int_{-\frac{\pi}{h}}^{0}t^{4m+3}\sin[t(x-jh)]\,\mathrm{d}t.
\end{aligned}$$

Now, through integrating by parts, the above formulas immediately lead to the expressions in (2.6). □

We remark that by setting $m=0$ in Lemma 2.1, we can straightforwardly get the Toeplitz matrices in (2.3)-(2.5).

About generating functions and eigenvalue bounds for the Toeplitz matrices $T^{(m)}$, $m=1,2,\ldots$, we have the following results.

Theorem 2.1. [14] *Let $T^{(m)}$ be the n-by-n Toeplitz matrices defined in* (2.1). *Then the following statements hold true:*

(i) its generating function is $f(\theta)=(\imath\theta)^m$;

(ii) if m is an odd positive number, i.e., $m=2p+1$, $T^{(m)}$ is a singular skew-symmetric matrix with eigenvalues $\imath\lambda_j$, where

$$-\pi^{2p+1}\leq\lambda_j\leq\pi^{2p+1};$$

(iii) if m is an even positive number, i.e., $m = 2p$, $T^{(m)}$ is a nonsingular symmetric matrix with eigenvalues $(-1)^p\lambda_j$, where $\lambda_j \in (0, \pi^{2p})$.

In particular, the eigenvalues $-\lambda_j^{(2)}$ of $T^{(2)}$ are bounded as

$$4\sin^2\left(\frac{\pi}{2n+2}\right) \le \lambda_j^{(2)} \le \pi^2.$$

3. Banded Toeplitz Preconditioners

For the Toeplitz matrices $T^{(1)}$ and $T^{(2)}$, in [12] and [13] the authors proposed to use the banded Toeplitz matrices $B^{(1)}$ and $B^{(2)}$, defined by the tridiagonal matrices

$$B^{(1)} = \text{tridiag}\left[\frac{1}{2}, 0, -\frac{1}{2}\right] \quad \text{and} \quad B^{(2)} = \text{tridiag}[1, -2, 1], \tag{3.1}$$

as their preconditioners, respectively. Theoretical analyses and numerical experiments showed that these preconditioners possess satisfactory algebraic properties and high computational efficiency. Following this approach, in general, we construct the following banded Toeplitz matrices $B^{(m)}$ as the preconditioners for the Toeplitz matrices $T^{(m)}$:

(a) if m is an odd positive number, i.e., $m = 2p+1$, then $B^{(m)}$ is a Toeplitz matrix generated by $g(\theta) = \imath\sin\theta(2\cos\theta - 2)^p$;

(b) if m is an even positive number, i.e., $m = 2p$, then $B^{(m)}$ is a Toeplitz matrix generated by $g(\theta) = (2\cos\theta - 2)^p$.

For example, the banded Toeplitz matrices $B^{(3)}$ and $B^{(4)}$ are penta-diagonal matrices and given by

$$B^{(3)} = \text{pentadiag}\left[\frac{1}{2}, -1, 0, 1, -\frac{1}{2}\right] \quad \text{and} \quad B^{(4)} = \text{pentadiag}\,[1, -4, 6, -4, 1], \tag{3.2}$$

respectively. We remark that the preconditioners $B^{(m)}$ are banded Toeplitz matrices with bandwidth being $m+1$ when m is even and being $m+2$ when m is odd.

The following theorem describes estimates about bounds on the eigenvalues of the banded Toeplitz matrices $B^{(m)}$.

Theorem 3.1. *Let $B^{(m)}$ be the n-by-n banded Toeplitz matrices defined in (a)-(b). Then the following statements hold true:*

(i) if m is an odd positive number, i.e., $m = 2p+1$, $B^{(m)}$ is a singular skew-symmetric matrix with eigenvalues $\imath\lambda_j$, where

$$-\frac{2^p(2p+1)^{p+\frac{1}{2}}}{(p+1)^{p+1}} < \lambda_j < \frac{2^p(2p+1)^{p+\frac{1}{2}}}{(p+1)^{p+1}};$$

(ii) if m is an even positive number, i.e., $m = 2p$, $B^{(m)}$ is a nonsingular symmetric matrix with eigenvalues $(-1)^p\lambda_j$, where

$$0 < \lambda_j < 4^p.$$

Proof. We first verify the validity of (i). Because the generating function of $B^{(m)}$ $(m = 2p+1)$ is

$$g(\theta) = \imath \sin\theta(2\cos\theta - 2)^p \equiv \imath\widetilde{g}(\theta), \qquad \theta \in [-\pi, \pi],$$

where

$$\widetilde{g}(\theta) = \sin\theta(2\cos\theta - 2)^p,$$

from [9, Theorem 1.11] we have

$$\min_{-\pi\leq\theta\leq\pi} \widetilde{g}(\theta) < \min \operatorname{Im}(\lambda(B^{(m)})) \leq \max \operatorname{Im}(\lambda(B^{(m)})) < \max_{-\pi\leq\theta\leq\pi} \widetilde{g}(\theta),$$

where $\operatorname{Im}(\cdot)$ denotes the imaginary part of the corresponding complex number. By directly calculating the minimum and the maximum values of $\widetilde{g}(\theta)$, we obtain (i).

Analogously, because the generating function of $B^{(m)}$ $(m = 2p)$ is $(2\cos\theta - 2)^p$, from [9, Theorem 1.11] again we have

$$\begin{aligned} 0 = &\min_{-\pi\leq\theta\leq\pi} (2 - 2\cos\theta)^p < (-1)^p \lambda_{\min}(B^{(m)}) \\ \leq &(-1)^p \lambda_{\max}(B^{(m)}) < \max_{-\pi\leq\theta\leq\pi} (2 - 2\cos\theta)^p = 4^p, \end{aligned}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimal and the maximal eigenvalues of the corresponding Hermitian matrix, respectively. This shows that (ii) is true. □

Specially, for $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ and $B^{(4)}$, we have the following sharper bounds about their eigenvalues.

Theorem 3.2. *Let $B^{(m)}$ $(m = 1, 2, 3, 4)$ be the n-by-n banded Toeplitz matrices defined in* (3.1)-(3.2). *Then the following statements hold true:*

(i) $B^{(1)}$ is a skew-symmetric matrix and its eigenvalues $\imath\lambda_j^{(1)}$ satisfy

$$-\cos\left(\frac{\pi}{n+1}\right) \leq \lambda_j^{(1)} \leq \cos\left(\frac{\pi}{n+1}\right);$$

(ii) $B^{(2)}$ is a symmetric negative-definite matrix and its eigenvalues $-\lambda_j^{(2)}$ satisfy

$$4\sin^2\left(\frac{\pi}{2n+2}\right) \leq \lambda_j^{(2)} \leq 4\cos^2\left(\frac{\pi}{2n+2}\right);$$

(iii) $B^{(3)}$ is a skew-symmetric matrix and its eigenvalues $\imath\lambda_j^{(3)}$ satisfy

$$-\frac{3\sqrt{3}}{2} < \lambda_j^{(3)} < \frac{3\sqrt{3}}{2};$$

(iv) $B^{(4)}$ is a symmetric positive-definite matrix and its eigenvalues $\lambda_j^{(4)}$ satisfy

$$16\sin^4\left(\frac{\pi}{2n+2}\right) \leq \lambda_j^{(4)} < 16.$$

Proof. (i) and (ii) can be found in [8, pp. 65-67]; see also [3, 7]. By making use of Theorem 3.1, we can obtain (iii) and the upper bound in (iv). It turns out that we only need to demonstrate the validity of the lower bound in (iv).

To this end, we observe that

$$B^{(4)} = (B^{(2)})^2 + W,$$

with

$$W = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

being a symmetric positive-semidefinite matrix. From (ii) we know that the minimal eigenvalue of $(B^{(2)})^2$ is $16\sin^4[\pi/(2n+2)]$. Therefore, according to Weyl's Monotonicity Theorem [15, pp. 101-102] we see that the minimal eigenvalue of $B^{(4)}$ is bounded below by $16\sin^4[\pi/(2n+2)]$. $\square$

4. Some Useful Bounds

In this section, we will demonstrate that the spectra of $(B^{(m)})^{-1}T^{(m)}$ are uniformly bounded by constants when m is even, and derive other bounds about the generalized Rayleigh quotients associated with the matrix sequences $\{B^{(m)}\}$ and $\{T^{(m)}\}$ when m is odd. These estimates are useful in obtaining eigenvalue bounds for the preconditioned matrices when the generalized Bendixson theorem is employed; see [2–4].

Theorem 4.1. *Let $T^{(m)}$ and $B^{(m)}$ be the Toeplitz matrices defined in* (2.1) *and (a)-(b) in Section 3, $f_m(\theta)$ and $g_m(\theta)$ be the generating functions of $T^{(m)}$ and $B^{(m)}$ respectively, and D is a given positive diagonal matrix. Then, for all $x \neq 0$, the following statements hold true:*

(i) if m is an even positive number, i.e., $m = 2p$,

$$1 \le \frac{x^*T^{(m)}x}{x^*B^{(m)})x} \le \frac{\pi^{2p}}{4^p};$$

(ii) if m is an odd positive number, i.e., $m = 2p+1$,

$$\max_{x\neq 0}\left\{\frac{x^*T^{(m)}(T^{(m)})^*x}{x^*(T^{(m-1)}+D)x}\right\} < \pi^{2p+2} \quad \textit{and} \quad \max_{x\neq 0}\left\{\frac{x^*B^{(m)}(B^{(m)})^*x}{x^*(B^{(m-1)}+D)x}\right\} < \frac{(4p+4)^{p+1}}{(p+2)^{p+2}}$$

when p is even, and

$$\max_{x\neq 0}\left\{\frac{x^*T^{(m)}(T^{(m)})^*x}{x^*(T^{(m+1)}+D)x}\right\} < \pi^{2p} \quad \textit{and} \quad \max_{x\neq 0}\left\{\frac{x^*B^{(m)}(B^{(m)})^*x}{x^*(B^{(m+1)}+D)x}\right\} < \frac{(4p)^p}{(p+1)^{p+1}}$$

when p is odd.

Here $(\cdot)^*$ denotes the conjugate transpose of either a vector or a matrix.

Proof. We first demonstrate (i). When m is an even positive number, recall that the generating functions of $T^{(m)}$ and $B^{(m)}$ $(m = 2p)$ are $(-\theta^2)^p$ and $(2\cos\theta - 2)^p$, respectively. Hence, for any $x \neq 0$ we have

$$\frac{x^*T^{(m)}x}{x^*B^{(m)}x} = \frac{\int_{-\pi}^{\pi} |\sum_{j=1}^{n} x_j e^{-\imath j\theta}|^2(-\theta^2)^p \,\mathrm{d}\theta}{\int_{-\pi}^{\pi} |\sum_{j=1}^{n} x_j e^{-\imath j\theta}|^2(2\cos\theta - 2)^p \,\mathrm{d}\theta}. \tag{4.1}$$

Define

$$\widetilde{f}(\theta) = \begin{cases} \dfrac{(\theta^2)^p}{(2-2\cos\theta)^p}, & \text{for} \quad \theta \in [-\pi,\pi]\setminus\{0\}, \\ 1, & \text{for} \quad \theta = 0. \end{cases}$$

Then by straightforward computations we obtain

$$1 \leq \widetilde{f}(\theta) \leq \frac{\pi^{2p}}{4^p}, \qquad \forall\, \theta \in [-\pi,\pi].$$

Now, by making use of (4.1) we immediately get

$$1 \leq \frac{x^*T^{(m)}x}{x^*B^{(m)}x} \leq \frac{\pi^{2p}}{4^p}.$$

We now turn to verify the validity of (ii). When p is an even number, from Theorem 2.1 we know that $T^{(m)}$ $(m = 2p+1)$ is a skew-symmetric Toeplitz matrix with its generating function in the Wiener class. By making use of Theorems 3.1 and 3.3 in [5], we see that for any $\epsilon > 0$ there exist a positive semidefinite matrix R_m of fixed rank and a matrix E_m of small norm such that $\|E_m\|_2 < \epsilon$ and

$$T^{(m)}(T^{(m)})^* + R_m + E_m = \widehat{T}^{(m)},$$

where $\widehat{T}^{(m)}$ is the Toeplitz matrix generated by the positive function $|f_m(\theta)|^2$. Because $T^{(m-1)}$ $(m = 2p+1)$ is positive definite when p is even, we have

$$\frac{x^*R_mx}{x^*(T^{(m-1)}+D)x} \geq 0 \quad \text{and} \quad \left|\frac{x^*E_mx}{x^*(T^{(m-1)}+D)x}\right| \leq \epsilon, \qquad \forall x \neq 0.$$

It then follows from the above matrix decompositions that

$$\max_{x\neq 0}\left\{\frac{x^*T^{(m)}(T^{(m)})^*x}{x^*(T^{(m-1)}+D)x}\right\} < \max_{x\neq 0}\left\{\frac{x^*\widehat{T}^{(m)}x}{x^*(T^{(m-1)}+D)x}\right\} + \epsilon.$$

Since ϵ is arbitrary, this inequality readily implies

$$\begin{aligned} \max_{x\neq 0}\left\{\frac{x^*T^{(m)}(T^{(m)})^*x}{x^*(T^{(m-1)}+D)x}\right\} &\leq \max_{x\neq 0}\left\{\frac{x^*\widehat{T}^{(m)}x}{x^*(T^{(m-1)}+D)x}\right\} < \max_{-\pi\leq\theta\leq\pi}\left\{\frac{|f_m(\theta)|^2}{f_{m-1}(\theta)}\right\} \\ &= \max_{-\pi\leq\theta\leq\pi}\frac{\theta^{4p+2}}{(-1)^p\theta^{2p}} = \pi^{2p+2}. \end{aligned}$$

In addition, when p is an even number, from Theorem 3.2 we know that $B^{(m)}$ $(m = 2p+1)$ is a skew-symmetric Toeplitz matrix. By making use of [5, Theorem 3.1] again we see that there exists a positive semidefinite matrix F_m of fixed rank such that

$$B^{(m)}(B^{(m)})^* + F_m = \widehat{B}^{(m)},$$

where $\widehat{B}^{(m)}$ is the Toeplitz matrix generated by the positive function $|g_m(\theta)|^2$. Because $B^{(m-1)}$ $(m = 2p+1)$ is positive definite when p is even, we have

$$\frac{x^* F_m x}{x^* B^{(m-1)} x} \geq 0, \qquad \forall x \neq 0.$$

It then follows from the above matrix decompositions that

$$\begin{aligned}\max_{x\neq 0}\left\{\frac{x^* B^{(m)}(B^{(m)})^* x}{x^*(B^{(m-1)}+D)x}\right\} &\leq \max_{x\neq 0}\left\{\frac{x^* \widehat{B}^{(m)} x}{x^*(B^{(m-1)}+D)x}\right\} < \max_{-\pi\leq\theta\leq\pi}\left\{\frac{|g_m(\theta)|^2}{g_{m-1}(\theta)}\right\} \\ &= \max_{-\pi\leq\theta\leq\pi}\frac{\sin^2\theta(2\cos\theta-2)^{2p}}{(2\cos\theta-2)^p} = \frac{(4p+4)^{p+1}}{(p+2)^{p+2}}.\end{aligned}$$

By similar arguments, we can obtain the upper bounds in (ii) when p is an odd number. □

Remark 4.1. Theorem 4.1 (i) can be regarded as a special case in [6, Theorem 2]. Theorem 4.1 (ii) can be used to estimate generalized Rayleigh quotients of the Hermitian and skew-Hermitian parts of the coefficient matrices obtaining from the sinc discretization for linear ordinary or partial differential equations, as the Hermitian parts of the coefficient matrices are combinations of $T^{(m)}$ (m is even) and diagonal matrices D; see [1–3]. Because preconditioners for the coefficient matrices can be formed as the same structure as the coefficient matrices, we can also employ Theorem 4.1 (ii) to estimate generalized Rayleigh quotients of the Hermitian and skew-Hermitian parts of the corresponding structured preconditioners.

5. Numerical Examples

In this section, we will show the effectiveness of the banded Toeplitz preconditioners. Here, we first apply Krylov subspace method, incorporated with the banded Toeplitz preconditioner $B^{(m)}$, to the system of linear equations

$$(B^{(m)})^{-1}T^{(m)}x = (B^{(m)})^{-1}b, \tag{5.1}$$

where m is an even number. We note that when $m = 4$, $T^{(4)}$ and $B^{(4)}$ are symmetric positive-definite Toeplitz matrices and conjugate gradient (CG) method can be used to solve (5.1). When $m = 2$ or $m = 6$, $T^{(m)}$ and $B^{(m)}$ are symmetric negative-definite Toeplitz matrices and we multiply -1 on these matrices so that CG method can also be used.

Table 1 lists the number of iteration steps required for convergence when CG method is applied to solve the linear system (5.1). In actual computations, we choose the right-hand-side vector b such that the exact solution of the linear system (5.1) is ones. All runs are started from the initial vector $x^{(0)} = 0$ and terminated if the current residual satisfies $||r^{(j)}||_2/||r^{(0)}||_2 \leq 10^{-6}$. In the table, I denotes no preconditioner is used.

From Table 1, we see that if no preconditioner is used, CG method converges very slowly and the number of iteration steps increases very fast when n is growing. However, when the banded Toeplitz preconditioner $B^{(m)}$ is used, the preconditioned CG method converges in much less iteration steps and the number of iteration steps keeps almost invariant when n is becoming large. Hence, the banded Toeplitz preconditioner $B^{(m)}$ is very effective in accelerating the convergence rates of CG method when m is even.

Note that when m is an odd number, $T^{(m)}$ and $B^{(m)}$ are singular skew-symmetric matrices. So we can not solve the system of linear equations as (5.1). However, these matrices are

Table 5.1: Numbers of iterations for the linear system (5.1)

n	$T^{(2)}$		$T^{(4)}$		$T^{(6)}$	
	I	$B^{(2)}$	I	$B^{(4)}$	I	$B^{(6)}$
16	8	7	9	7	13	8
32	16	9	28	11	49	12
64	37	10	98	13	161	16
128	82	10	384	15	620	20
256	176	10	1093	16	2243	23
512	370	10	3595	16	6404	24

Table 5.2: Numbers of iterations for the linear system (5.3)

N	n	GMRES		BiCGSTAB	
		I	P	I	P
16	33	22	14	46	10
32	65	51	19	152	12
64	129	114	23	856	15
128	257	239	26	$*$	17
256	513	494	27	$*$	17
512	1025	1004	28	$*$	18

used in the systems of linear equations arising from sinc discretizations of ordinary and partial differential equations.

Now we consider a fourth-order linear homogeneous boundary-value problem in [11]

$$\begin{cases} \dfrac{\mathrm{d}^4}{\mathrm{d}x}u(x) = \dfrac{9}{16}x^{-5/2}(1-x)^{-5/2}, \\ u(0) = u(1) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}x}u(0) = \dfrac{\mathrm{d}}{\mathrm{d}x}u(1) = 0. \end{cases} \tag{5.2}$$

After discretizing the problem (5.2) by sinc-collocation method, we obtain a system of linear equations as follows

$$A\mathbf{u} = \mathbf{b}, \tag{5.3}$$

where

$$A = T^{(4)} + D^{(3)}T^{(3)} + D^{(2)}T^{(2)} + D^{(1)}T^{(1)} + D^{(0)}$$

is a $n \times n$ matrix with $D^{(i)}$ being diagonal matrices; see [11] for more details. We construct the following structured preconditioner P for the coefficient matrix A

$$P = B^{(4)} + D^{(3)}B^{(3)} + D^{(2)}B^{(2)} + D^{(1)}B^{(1)} + D^{(0)}.$$

Then we apply GMRES and BiCGSTAB, incorporated with the preconditioner P, to the system of linear equations obtained from the sinc discretization of the problem (5.2).

Table 2 lists the number of iteration steps required for convergence when GMRES and BiCGSTAB methods are used to solve the linear system (5.3). In our tests, the initial guess is taken to be ones and the iteration process is terminated once the current iteration satisfies $||r^{(j)}||_2/||r^{(0)}||_2 \leq 10^{-6}$. In the table, we use "$*$" to indicate that the iteration method does not converge within 10000 iterations, I to represent the iteration method with no preconditioner, and P to denote the iteration method with the structured preconditioner P.

From Table 2, we see that if no preconditioner is used, GMRES converges very slowly and the number of iteration steps increases approximately like n. BiCGSTAB converges more slowly than GMRES for this problem, and it even fails to solve the linear system (5.3) when $n \geq 257$. However, when the preconditioner P is used, the preconditioned GMRES and the preconditioned BiCGSTAB can successfully compute satisfactory approximations to the exact solutions of the problem (5.2), and both methods converge in much less iteration steps. Hence, for this example the structured preconditioner P is very effective in accelerating the convergence rates of GMRES and BiCGSTAB. Therefore, the preconditioner $B^{(m)}$ is a good approximation to the Toeplitz matrix $T^{(m)}$.

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