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THE ULTRACONVERGENCE OF EIGENVALUES FOR BI-QUADRATIC FINITE ELEMENTS*

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Abstract

The classical eigenvalue problem of the second-order elliptic operator is approximated with bi-quadratic finite element in this paper. We construct a new superconvergent function recovery operator, from which the $O(h^8|\ln h|^2)$ ultraconvergence of eigenvalue approximation is obtained. Numerical experiments verify the theoretical results.

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1. Introduction

Many post-processing techniques have been proposed for the finite element method, and they are widely used in scientific and engineering application. For the literature, readers are referred to [1, 3, 4, 29, 30, 33], and references therein.

One of the post-processing methods is the gradient recovery technique [12,14,15,32,33]. Recently, this technique has been used to improv the eigenvalue approximation by the linear finite element method [9, 17, 20]. It turned out that the convergent rate can be doubled in the most favorable situation. In this paper, we design a function recovery operator for the quadratic finite element method to enhance the eigenvalue approximation. Due to the complexity nature of the recovery operator in higher-order situation, this extension is nevertheless non-trivial as we may see from the later sections.

For the quadratic finite element method, Lin and Yang [13] discovered and proved that derivatives of bi-quartic interpolation I_4u_h of the solution of biquadratic element have 4th-order super-convergence, and the Rayleigh quotient of I_4u_h has eight-order super-convergence.

One of the strategies in eigenvalue enhancement is to use the Rayleigh quotient as e.g., in [9, 13, 18, 26]. This is also the strategy we will use in this work. The key idea is to replace the finite element gradient (or its resultants) on the numerator of the Rayleigh quotient by the recovered gradient. At the same time, the denominator has to be changed in order to consistent with the recovered gradient. Therefore, a function value recovery is also required.

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The recovery technique is most effective under uniform meshes, or the meshes with regular refinement. In this work, we only concentrate on rectangular meshes, and leave the triangular case to a forthcoming paper. The new recovery operator proposed here works remarkably well for our purpose. As we shall see in Section 5, an $O(h^8|\ln h|^2)$ convergence rate for eigenvalue approximation is obtained.

Different from the linear element case, in which the function value recovery has no superconvergence [19], the function recovery operator constructed for quadratic element in our work is superconvergent.

The paper is organized as follows. Section 2 outlines the eigenvalue problems and their Galerkin approximation. In Section 3, the derivative recovery technique is introduced, and the ultraconvergence of derivatives is shown for the bi-quadratic finite element. Then we construct the function recovery operator and discuss its properties. The eigenvalue recovery is given in the Raylaigh quotient theme in Section 4, and the main results are proved in Section 5. Finally, numerical tests are provided in Section 6 to demonstrate the effectiveness of our method.

2. Problem and Its Galerkin Approximation

Consider the following classic eigenvalue problem of the Laplacian operator:

$$\begin{cases} Lu \equiv -\nabla \cdot (\mathbf{D}\nabla u) + cu = \lambda u, \quad x \in \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

where Ω is a bounded polygonal domain, $\mathbf{D}$ is a 2×2 positive definite matrix on Ω, and c is a sufficiently smooth function.

Let $V = H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Then the variational formulation of the problem (2.1) is to seek the eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ such that

$$a(u,v) = \lambda b(u,v),\ \forall\, v \in V, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega (\mathbf{D}\nabla u)\cdot \nabla v + cuv, \quad b(u,v) = \int_\Omega uv.$$

For simplicity, in this paper we shall discuss the case when $D = I$ and $c = 0$, then the bilinear $a(\cdot,\cdot)$ is bounded and V-elliptic, namely, there exist the constants M_1 and M_2 such that

$$\begin{aligned} |a(u,v)| &\leqslant M_1\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \\ a(u,u) &\geqslant M_2\|u\|_{1,\Omega}^2, \end{aligned} \quad \forall\, u, v \in V.$$

Let $\mathcal{T}_h$ be a quasi-uniform rectangulation of Ω. In this paper, the bi-quadratic finite element method is used to solve the problem (2.1). And let V_h denote the bi-quadratic finite element space, i.e.,

$$V_h = \Big\{ v \in C(\overline{\Omega}) : v|_\tau \in Q_2(\tau) \text{ for every rectangle element } \tau \in \mathcal{T}_h \Big\}.$$

And we denote $V_h^0 = V_h \bigcap V$. So the finite element approximation (λ_h, u_h) of (λ, u) in (2.2) can be computed as the following scheme: find $(\lambda_h, u_h) \in \mathbb{R} \times V_h^0$ such that

$$a(u_h, v) = \lambda_h (u_h, v),\ \forall\, v \in V_h^0. \tag{2.3}$$

Since $a(\cdot,\cdot)$ is symmetric, (2.1) has a countable sequence of real eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. Consequently, the corresponding form (2.3) has a finite sequence of eigenvalues $0 < \lambda_{1h} \leqslant \lambda_{2h} \leqslant \cdots \leqslant \lambda_{n_h h}$ where $n_h = \dim V_h^0$, and the inequalities $\lambda_{ih} \geqslant \lambda_i$ hold for $i = 1, 2, \cdots, n_h$. See [2,10,11] for more details.

For the convenience of our introduction, we adopt the Lagrange interpolation for our analysis, in which it is customary to employ the equidistributed set of interpolation points. While we denote the set of all nodes in $\mathcal{T}_h$ for the bi-quadratic and bi-4 degree finite elements by $\mathcal{N}_h$ and $\bar{\mathcal{N}}_h$, respectively. And we set $\mathcal{M}_h = \bar{\mathcal{N}}_h - \mathcal{N}_h$. In addition, let $\bar{V}_h$ be the finite element space of degree 4, then analogously denote $\bar{V}_h^0 = \bar{V}_h \bigcap V$. The location of all nodes in the closure of an element τ can be seen in Fig. 3.1.

In addition, $H^m(\Omega)$ is the usual Sobolev space of order m equipped with the norm $\|\cdot\|_{m,\Omega}$ and the semi-norm $|\cdot|_{m,\Omega}$; The space of all polynomials defined on Ω of total degree $\leqslant k$ is denoted by $P_k(\Omega)$, and the one of all bi-k degree polynomials on Ω is denoted by $Q_k(\Omega)$.

Now we quote the following two results. The first result is about an estimation for the interpolation of u on Ω.

Theorem 2.1. *Assume that $\mathcal{T}_h$ is a quasi-uniform rectangulation on Ω and $i_k u$ is the piecewise bi-k polynomial interpolation of $u \in H^{k+1}(\Omega)$. Then*

$$\|i_k u - u\|_{m,\Omega} \leqslant Ch^{k+1-m}\|u\|_{k+1,\Omega},\ m = 0, 1.$$

And another superconvergent result can be seen in [21] (P. 44) and in [31] (P. 169).

Theorem 2.2. *Assume that $\mathcal{T}_h$ is a quasi-uniform rectangulation on Ω and u_h is the bi-quadratic finite element approximation of $u \in W^{4,\infty}(\Omega) \bigcap H_0^1(\Omega)$. Then for every nodes $z \in \mathcal{N}_h$,*

$$|u_h(z) - u(z)| \leqslant Ch^4 |\ln h|\, \|u\|_{4,\infty,\Omega}.$$

3. The Function Recovery Operator

3.1. The derivative recovery and its superconvergence

For more than thirty years, an impressive amount of work has been done to study the post-processing techniques for the finite element solution. In 2005, Zhang and Naga [29] generalized the SPR technique [33] and proposed the polynomial preserving technique. By using it the superconvergence of the recovered derivatives is obtained for the linear finite element. And numerical tests indicate there exists the ultraconvergence of derivative for bi-quadratic finite element, which is used and important in this paper. Moreover, we can show the numerical result through a completely analogous analysis to the proof in Theorem 2.1 in [28].

Theorem 3.1. *Assume that $\mathcal{T}_h$ is a uniform rectangular meshes on Ω, and $G_h u_h \in [Q_2(\Omega)]^2$ is the recovered gradient of $u \in H_0^5(\Omega)$ by the PPR technique, then for each $z \in \mathcal{N}_h$,*

$$|G_h u_h(z) - \nabla u(z)| \leqslant Ch^4 |\ln h| \|u\|_{5,\Omega}, \tag{3.1}$$

Further,

$$\|G_h u_h - \nabla u\|_{0,\Omega} \leqslant Ch^4 |\ln h| \|u\|_{5,\Omega}. \tag{3.2}$$

3.2. The construction of the function recovery operator

We'll construct the function recovery operator $R_h : V_h \to \bar{V}_h$ to enhence the eigenvalue approximation in this part. To define the recovered function $R_h v$ of $v \in V_h$ in the space $\bar{V}_h$, as we know, it suffices to determine its function value at all nodal points in $\overline{\mathcal{N}}_h$. So $\forall\, z \in \overline{\mathcal{N}}_h$,

1. when $z_i \in \mathcal{N}_h$, set $R_h v(z_i) = v(z_i)$;

2. when $p_i \in \mathcal{M}_h = \overline{\mathcal{N}}_h - \mathcal{N}_h$, we assume p_i is in the closure of some an element $\tau \in \mathcal{T}_h$, then we define $R_h v(p_i)$ as the following two steps: Firstly, find a polynomial $Q_5'(x, y)$ of degree 5 on $\bar{\tau}$ such that $Q_5'(z_i) = v(z_i)$ and $\nabla Q_5'(z_i) = G_h v(z_i)\,(i = 1, 2, \cdots, 9)$, where $z_i \in \mathcal{N}_h \cap \bar{\tau}$ (see Fig. 3.1); Secondly, set $R_h v(p_i) = Q_5'(p_i)$.

It is obvious that the above defined function $R_h v$ is continuous.

In addition, $R_h v(p_i)$ can be obtained more simply and directly in practical implement. We first note a fact that there exist some two points in $\mathcal{M}_h$ such that they must be always in some a line segment with the ends which are in $\mathcal{N}_h$. For example, in Fig. 3.1, in an element p_{15} and p_{16} are in the line segment $\overrightarrow{z_4 z_3}$ and z_7 is its mid-point, p_{11} and p_6 are in the line segment $\overrightarrow{z_4 z_2}$ and z_9 is its mid-point, and p_8 and p_9 are in the line segment $\overrightarrow{z_8 z_6}$ and z_9 is its mid-point, etc. Generally, it is sufficient to let $q_1, q_2 \in \mathcal{M}_h$ be in a line segment l with ends e_1 and e_2, and let e_m be the mid-point of the line segment l, namely, $q_1 = \frac{1}{4}(3e_1 + e_2)$, $q_2 = \frac{1}{4}(e_1 + 3e_2)$, $e_m = \frac{1}{2}(e_1 + e_2)$. On the other hand, restricted to the line l, $Q_5'(x, y)$ is a one-variable polynomial of degree 5, we denote $P_5(l) = Q_5'(x, y)|_l$, Thus, $R_h v(q_i) = P_5(q_i)\,(i = 1, 2)$. Then the recovered values $R_h v(q_i)\,(i = 1, 2)$ follow directly computing.

$$R_h v(q_1) = \frac{45v(e_1) + 72v(e_m) + 11v(e_2)}{128} + \frac{\vec{l} \cdot (9G_h v(e_1) - 36G_h v(e_m) - 3G_h v(e_2))}{256}, \quad (3.3a)$$

$$R_h v(q_2) = \frac{11v(e_1) + 72v(e_m) + 45v(e_1)}{128} + \frac{\vec{l} \cdot (3G_h v(e_1) + 36G_h v(e_m) - 9G_h v(e_2))}{256}. \quad (3.3b)$$

For simplicity, we rewrite (3.3) into the following form.

$$R_h v(q_1) = C_1 v(E) + C_2(\vec{l} \cdot G_h v(E)), \quad (3.4a)$$

$$R_h v(q_2) = C_3 v(E) + C_4(\vec{l} \cdot G_h v(E)). \quad (3.4b)$$

where

$$C_1 = \frac{1}{128}[45\ 72\ 11], \quad C_2 = \frac{1}{256}[9\ \ -36\ \ -3],$$
$$C_3 = \frac{1}{128}[11\ 72\ 45], \quad C_4 = \frac{1}{256}[3\ 36\ \ -9], \qquad E = [e_1\ e_m\ e_2]^T.$$

3.3. The properties of the function recovery operator

Proposition 3.2. *The function recovery operator $R_h : V_h \to \bar{V}_h$ defined above satisfies the following properties:*

1). *$\forall\, \tau \in \mathcal{T}_h$ and $v \in V_h, R_h v|_\tau$ is obtained by using $v|_{E_\tau}$, where E_τ is the union of a finite sequence of elements around τ;*

2). *Assume that $u \in W^{5,\infty}(\Omega) \bigcap H^5(\Omega), \mathcal{T}_h$ is a uniform rectangulation on Ω and u_h is the bi-quadratic finite element approximation. Then, for any $z_i \in \overline{\mathcal{N}}_h$, there exist the following estimators.*

$$|R_h u_h(z_i) - u(z_i)| \leqslant Ch^4 |\ln h| \Big(\|u\|_{5,\infty,\Omega} + \|u\|_{5,\Omega} \Big), \quad (3.5a)$$

$$|\nabla(R_h v - v)(z_i)| \leqslant Ch^4 |\ln h| \Big(\|u\|_{5,\infty,\Omega} + \|u\|_{5,\Omega} \Big). \quad (3.5b)$$

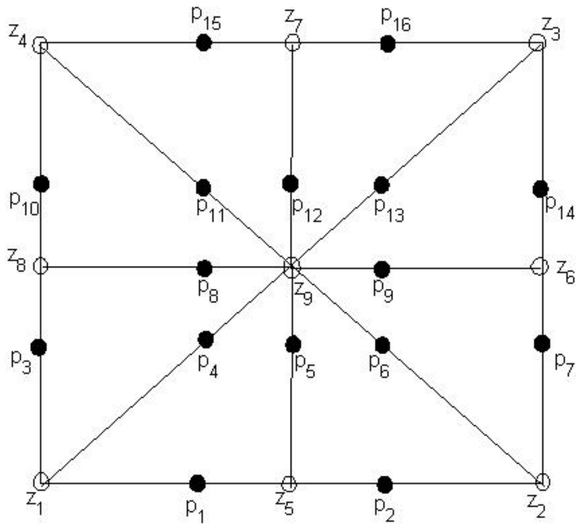

Fig. 3.1. Location of all nodes in an element τ for finite element space $\bar{V}_h$. $\circ$ denote the original nodes for finite element space V_h, i.e., $z_j \in \mathcal{N}_h$. $\bullet$ denote the additional nodes, i.e., $p_j \in \mathcal{M}_h$.

3). $\|R_h v\|_{0,\tau} \leqslant C\|v\|_{0,E_\tau}$ $\forall\, v \in V_h, \tau \in \mathcal{T}_h$, *where* $C > 0$ *is a constant independent of* h*;*
4). $\|R_h v\|_{0,\Omega} \geqslant C\|v\|_{0,\Omega}$, *where* $C > 0$ *is a constant independent of* h.

Proof. 1). By the definition of $G_h v$, it is easy to see that $G_h v(z)$ $\forall z \in \bar{\tau} \bigcap \mathcal{N}_h$ is a linear combination of $v(z_i)$ for $z_i \in E_\tau \bigcap \mathcal{N}_h$. Furthermore, so is $R_h v(z)$ $\forall z \in \bar{\tau} \bigcap \mathcal{N}_h$. Thus 1) holds.

2). If $u \in W^{5,\infty}(\Omega) \bigcap H^5(\Omega)$, and set $G_h u = \nabla u$, then we can generalize the operator R_h to the space $W^{5,\infty}(\Omega) \bigcap H^5(\Omega)$. Obviously, $R_h u = u$ holds for any polynomials $u \in Q_4(\Omega)$, so by Bramble-Hilbert Lemma we have

$$|u - R_h u|_{m,\infty} \leqslant Ch^{5-m}\|u\|_{5,\infty}, \quad m = 0, 1. \tag{3.6}$$

When $z_i \in \mathcal{N}_h$, $R_h u_h(z_i) = u_h(z_i)$. Then by Theorem 2.2,

$$|R_h u_h(z_i) - u(z_i)| = |u_h(z_i) - u(z_i)| \leqslant Ch^4 |\ln h| \|u\|_{4,\infty},$$

i.e., (3.5a) holds. When $p_1, p_2 \in \mathcal{M}_h$, as discussed above, let $p_1, p_2 \in l$ where l is a line segment with the ends $z_1, z_2 \in \mathcal{N}_h$, $z_m \in \mathcal{N}_h$ is its mid-points, and $p_1 = \frac{1}{4}(3z_1 + z_2)$, $p_2 = \frac{1}{4}(z_1 + 3z_2)$. Moreover, we denote $Z = [z_1, z_m, z_2]^T$. Then

$$\begin{aligned}
&|R_h u_h(p_1) - u(p_1)| \\
=&|C_1 u_h(Z) + C_2(\vec{l} \cdot G_h u_h(Z)) - u(p_1)| \\
\leqslant&|C_1(u_h(Z) - u(Z)) + C_2(\vec{l} \cdot (G_h u_h(Z) - \nabla u(Z)))| \\
&\quad + |C_1 U + C_2(\vec{l} \cdot \nabla u(Z)) - u(p_1)| \\
\equiv&E_1 + E_2.
\end{aligned} \tag{3.7}$$

By Theorem 2.2, Theorem 3.1 and (3.6),we have

$$|R_h u_h(p_1) - u(p_1)| \leqslant Ch^4 |\ln h| \Big(\|u\|_{5,\infty,\Omega} + \|u\|_{5,\Omega} \Big).$$

By the same method we may prove

$$|R_h u_h(p_2) - u(p_2)| \leqslant Ch^4 |\ln h| \Big(\|u\|_{5,\infty,\Omega} + \|u\|_{5,\Omega} \Big).$$

Summarizing the above analysis, (3.5a) holds. Similarly to the above process, (3.5b) holds.

3). From the definition of G_h and R_h, it is obvious that for $z_i \in \bar{\tau} \bigcap \mathcal{N}_h$ $G_h v(z_i)$ is a linear combination of $\{v(z_j) : z_j \in E_\tau \bigcap \mathcal{N}_h\}$ and $R_h v(p_i)$ is a linear combination of

$$\Big\{ v(z_1), v(z_2), v(z_m), \vec{l} \cdot G_h v(z_1), \vec{l} \cdot G_h v(z_2), \vec{l} \cdot G_h v(z_m) : \\ z_j \in \bar{\tau} \bigcap \mathcal{N}_h, \, \vec{l} = \overrightarrow{z_1 z_2}, \, p_i = \frac{1}{4}(z_1 + 3z_2)\} \,\text{or}\, p_i = \frac{1}{4}(3z_1 + z_2) \Big\}.$$

So for $z_i \in \bar{\tau} \bigcap \mathcal{N}_h, R_h v(z_i)$ is a linear combination of $\{v(z_j) : z_j \in E_\tau \bigcap \mathcal{N}_h\}$, thus

$$|R_h v(z_i)| \leqslant C \max \Big\{ |v(z_j)| : z_j \in E_\tau \bigcap \mathcal{N}_h \Big\} \leqslant C \|v\|_{\infty, E_\tau}.$$

Hence,

$$\|R_h v\|_{\infty,\tau} \leqslant C \|v\|_{\infty, E_\tau}.$$

By the equivalence of the norms on the finite dimensional space, 3) holds. Moreover, 4) follows from the similar reason as 3). $\square$

4. The Recovery of Eigenvalues

In [18], Naga and Zhang recovered eigenvalues by the Rayleigh quotient. Here we still adopt the method to obtain the recovered eigenvalues $\hat{\lambda}_h$, i.e.,

$$\hat{\lambda}_h = \frac{a(R_h u_h, R_h u_h)}{b(R_h u_h, R_h u_h)}. \tag{4.1}$$

To prove the ultraconvergence of $\hat{\lambda}_h$ the following lemma is important [2].

Lemma 4.1. *Let (λ, u) be a solution of (2.1) and $a(\cdot,\cdot)$ symmetric. Then for any $w \in V \setminus \{0\}$, there holds*

$$\frac{a(w,w)}{b(w,w)} - \lambda = \frac{a(w-u, w-u)}{b(w,w)} - \lambda \frac{b(w-u, w-u)}{b(w,w)}. \tag{4.2}$$

5. The Proof of the Main Results

Theorem 5.1. *Assume that $u \in H_0^1(\Omega) \bigcap W^{5,\infty}(\Omega)$, $\mathcal{T}_h$ is a quasi-uniform rectangulation over Ω, and $R_h u_h$ is the above recovered function of the bi-quadratic finite element u_h. Then*

$$\|u - R_h u_h\|_{0,\Omega} \leqslant Ch^4 |\ln h| \|u\|_{5,\infty,\Omega}, \tag{5.1a}$$

$$\|u - R_h u_h\|_{1,\Omega} \leqslant Ch^4 |\ln h| \|u\|_{5,\infty,\Omega}. \tag{5.1b}$$

Proof. Let τ be any an element, and $i_4 u$ be the piecewise interpolation of degree 4 of function u on $\mathcal{T}_h$. From Theorem 2.1 and

$$\|u - R_h u_h\|_{0,\Omega} \leqslant \|u - i_4 u\|_{0,\Omega} + \|i_4 u - R_h u_h\|_{0,\Omega},$$

it is sufficient to show that $\|i_4u - R_hu_h\|_{0,\Omega} \leqslant Ch^4(\|u\|_{5,\Omega} + \|u\|_{4,\infty,\Omega})$. On the other hand, by Proposition 3.2 2) we have

$$\begin{aligned}\|i_4u - R_hu_h\|_{0,\Omega}^2 &= \sum_{\tau\in\mathcal{T}_h} \|i_4u - R_hu_h\|_e^2\\ &= \sum_{\tau\in\mathcal{T}_h} \int_\tau \left(\sum_{j=1}^{25}(u(z_j) - R_hu_h(z_j))l_j(x,y)\right)^2 \mathrm{d}x\mathrm{d}y\\ &\leqslant \max_{z_j\in\overline{\mathcal{N}}_h} |u(z_j) - R_hu_h(z_j)|^2 \sum_{\tau\in\mathcal{T}_h} \int_\tau \left(\sum_{j=1}^{25}|l_j(x,y)|\right)^2 \mathrm{d}x\mathrm{d}y\\ &\leqslant Ch^8|\ln h|^2\|u\|_{5,\infty,\Omega}^2,\end{aligned}$$

which is the desired results in (5.1a).

Now we prove (5.1b). Let $i_3\left(\frac{\partial u}{\partial x}\right)$ be the piecewise interpolation of degree 3 of function $\frac{\partial u}{\partial x}$ on $\mathcal{T}_h$, and the set of the interpolation nodes is $\hat{\mathcal{N}}_h$. Then for every element $\tau \in \mathcal{T}_h$,

$$\begin{aligned}i_3\left(\frac{\partial u}{\partial x}\right)(x,y) &= \sum_{z_j\in\bar\tau\bigcap\hat{\mathcal{N}}_h} \frac{\partial u}{\partial x}(z_j)l_j(x,y),\ (x,y)\in\bar\tau,\\ \frac{\partial R_hu_h}{\partial x}(x,y) &= \sum_{z_j\in\bar\tau\bigcap\hat{\mathcal{N}}_h} \frac{\partial R_hu_h}{\partial x}(z_j)l_j(x,y),\ (x,y)\in\bar\tau.\end{aligned}$$

So by (3.5b),

$$\begin{aligned}\left\|\frac{\partial R_hu_h}{\partial x} - i_3(\frac{\partial u}{\partial x})\right\|_{0,\Omega}^2 &= \sum_{\tau\in\mathcal{T}_h} \int_\tau \left(\sum_{j=1}^{16}(\frac{\partial R_hu_h}{\partial x}(z_j) - \frac{\partial u}{\partial x}(z_j))l_j(x,y)\right)^2 \mathrm{d}x\mathrm{d}y\\ &\leqslant \max_{z_j\in\hat{\mathcal{N}}_h} |(\tfrac{\partial R_hu_h}{\partial x} - \tfrac{\partial u}{\partial x})(z_j)| \sum_{\tau\in\mathcal{T}_h} \int_\tau \left(\sum_{j=1}^{16}|l_j(x,y)|\right)^2 \mathrm{d}x\mathrm{d}y\\ &\leqslant Ch^8|\ln h|^2\|u\|_{4,\infty,\Omega}^2,\end{aligned}$$

where $l_j(x,y)$ is the interpolating basic function with respecting to the point $z_j \in \bar\tau\bigcap\hat{\mathcal{N}}_h$. Hence, from Theorem 2.1 and the above inequality, we have

$$\begin{aligned}&\left\|\frac{\partial R_hu_h}{\partial x} - \frac{\partial u}{\partial x}\right\|_{0,\Omega}\\ \leqslant &\left\|\frac{\partial R_hu_h}{\partial x} - i_3\Big(\frac{\partial u}{\partial x}\Big)\right\|_{0,\Omega} + \left\|i_3\Big(\frac{\partial u}{\partial x}\Big) - \frac{\partial u}{\partial x}\right\|_{0,\Omega} \leqslant Ch^4|\ln h|\|u\|_{5,\infty,\Omega}.\end{aligned}$$

Similarly,

$$\left\|\frac{\partial R_hu_h}{\partial y} - \frac{\partial u}{\partial y}\right\|_{0,\Omega} \leqslant Ch^4|\ln h|\|u\|_{5,\infty,\Omega}.$$

So (5.1b) has been shown. □

Theorem 5.2. *Assume that $\mathcal{T}_h$ is a quasi-uniform rectangulation over Ω, $u \in H_0^1(\Omega)\bigcap W^{5,\infty}(\Omega)$, and $\hat\lambda_h$ is defined as (4.1), then*

$$\hat\lambda_h - \lambda \leqslant Ch^8|\ln h|^2\|u\|_{5,\infty,\Omega}.$$

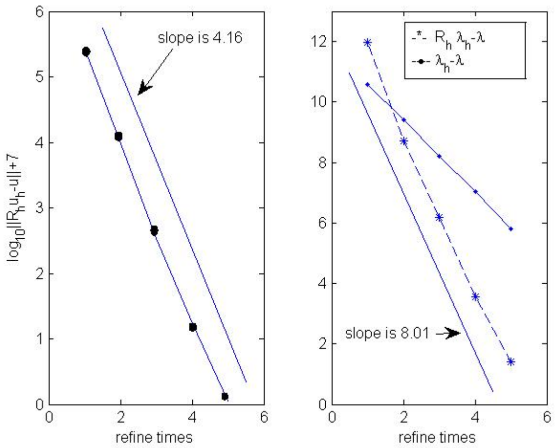

Fig. 6.1. (a) The convergent rate of $R_h u_h$. (b) The convergent rate of $\hat{\lambda}_h$.

Proof. By (4.2), we have

$$\hat{\lambda}_h - \lambda = \frac{a(u - R_h u_h, u - R_h u_h)}{\|R_h u_h\|^2} - \lambda \frac{\|u - R_h u_h\|^2}{\|R_h u_h\|^2}.$$

So the desirable inequality holds by using Theorem 5.1 and Proposition 3.2 4). $\square$

6. A Numerical Example

Consider the problem

$$\begin{cases} -\triangle u = \lambda u, & \text{in } \Omega = [-1,1]^2 \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

As is known, the eigenvalues of (6.1) are $\lambda_{mn} = \left(\frac{m\pi}{2}\right)^2 + \left(\frac{n\pi}{2}\right)^2$ and the corresponding eigenfunctions are

$$u_{mn}(x,y) = \sin\frac{m\pi(x+1)}{2}\sin\frac{n\pi(y+1)}{2} \qquad \text{for } m,n \in \mathbb{N}_+.$$

To compute the solution of (6.1), the bi-quadratic finite element method and the above eigenvalue recovery technique are used in this example, where the meshes $\mathcal{T}_h$ are uniform. Here, we recover the first eigenvalue approximation of the problem (6.1) by (4.1), and the convergence of the first recovered eigenvector $R_h u_h$ is given in Fig. 6.1(a), the convergence of the first eigenvalue approximation λ_h and its recovery $\hat{\lambda}_h$ is done in Fig. 6.1(b).

As we can see in Fig. 6.1, the convergent rate of $R_h u_h$ overruns 4, which confirms the estimates in Theorem 5.1, and the convergent rate of $\hat{\lambda}_h$ is close to 8, which demonstrates the results in Theorem 5.2.

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References

[1] M. Ainsworth J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, Wiley Interscience, New York, 2000.

[2] I. Babuska and J.E. Osborn, Eigenvalue problems, in: Handbook of Numerical Analysis Vol.II, Finite Element Methods (Part I), P.G. Ciarlet and J.L. Lions, eds, Elsevier, 1991, 641–792.

[3] I. Babuska and T. Strouboulis, The Finite Element Method and its Reliability, Oxford University Press, London, 2001.

[4] R.E. Bank, and J. Xu, Asymptotically exact a posteriori error estimators, Part I: Grid with superconvergence, *SIAM J. Numer. Anal.*, **41** (2003), 2294–2312.

[5] J.H. Bramble and J. Xu, A local post-processing technique for improving the accuracy in mixed finite element approximations, *SIAM J. Numer. Anal.*, **26** (1989), 1267–1275.

[6] S. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, 3rd edition, Springer-Verlag, New York, 2008.

[7] C. Carstensen, A unifying theory of a posteriori finite element error control, *Numer. Math.*, **1000** (2005), 617–637.

[8] A. Hannukainen, S. Korotov and M. Krizek, Nodal O(h4)-Superconvergence in 3D by Averaging Piecewise Linear, Bilinear, and Trilinear FE Approximation, *J. Comp. Math.*, **28**, (2010), 1–10.

[9] K. Kolman, A two-level method for nonsymmetric eigenvalue problems, *Acta Math. Appl. Sin. Engl. Ser.*, **21**:1, (2005), 1–12.

[10] J.R. Kuttler and V.G. Sigillito, Eigenvalues of the Laplacian in two dimensions, *SIAM Review,* **26**:2 (1984), 163–193

[11] M.P. Lebaud, Error Estimate in an isoparametric jinite ylement yigenvalue problem, *Math. Comp.*, **63**:207 (1994), 19–40.

[12] X.D. Li and N.-E. Wiberg, A posteriori error estimate by element patch postprocessing, adaptive analysis in energy and L^2 norms, *Comp. Struct.*, **53** (1994), 907–919.

[13] Q. Lin and Y.D. Yang, Interpolation and correction of finite elements. *Math Prac Theory*, **3** (1991), 17–28, (in Chinese).

[14] L.X. Meng and Q.D. Zhu, The ultraconvergence of derivative for bicubic finite element, *Comput. Methods Appl. Mech. Engrg.*, **196** (2007), 3771-3778.

[15] A. Naga and Z. Zhang, A posteriori error estimates based on polynomial preserving recovery, *SIAM J. Numer. Anal.*, **9** (2004), 1780–1800.

[16] A. Naga and Z. Zhang, The polynomial-preserving recovery for higher order finite element methods in 2D and 3D, *Discrete and Continuous Dynamical Systems Series B*, **5** (2005), 769–708.

[17] A. Naga, Z. Zhang, and A. Zhou, Enhancing eigenvalue approximation by gradient recovery, *SIAM J. Sci. Comput.*, **28** (2006), 1289–1300.

[18] A. Naga and Z. Zhang, Function value recovery and its application in linear finite elements, *SIAM J. Numer. Anal.*, **50**:1, (2012), 272–286.

[19] J.S. Ovall, Function, Gradient and Hessian recovery using quadratic edge-bump functions, *SIAM J. Numer. Anal.*, **5** (2007), 1064–1080.

[20] L. Shen and A. Zhou, A defect correction scheme for finite element eigenvalues with applications to quantum chemistry, *SIAM J. Sci. Comput.*, **28**:1 (2006), 321-3-38. (electronic).

[21] L.R. Wahlbin, Superconvergence in Galerkin Finite Element Methods, Lecture Notes in Mathematics, Vol. 1605, Springer, Berlin, 1995.

[22] J. Wang, A superconvergence analysis for finite element solutions by the least-squares surface ting on irregular meshes for smooth problems, *J. Math. Study*, **33** (2000), 229–243.

[23] N. E. Wiberg and X.D. Li, Superconvergent patch recovery of ite element solution and a posteriori error L^2 norm estimate, *Comm. Numer. Methods Engrg.*, **10** (1994), 313–320.

[24] H. Wu and Z. Zhang, Can we have superconvergent gradient recovery under adaptive meshes? *SIAM J. Numer. Anal.*, **45-4** (2007), 1701–1722.

[25] J. Xu and Z. Zhang, Analysis of recovery type a posteriori error estimators for mildly structured grids, *Math. Comp.*, **73** (2004), 1139–1152.

[26] J. Xu and A. Zhou, A two-grid discretization scheme for eigenvalue problems, *Math. Comp.*, **70** (2001), 17-25.

[27] J. Xu and A. Zhou, Local and parallel finite element algorithms for eigenvalue problems, *Acta Math. Appl. Sin. Engl. Ser.*, **18**:2 (2002), 185–200.

[28] Z. Zhang and R.C. Lin, Ultraconvergence of ZZ patch recovery at mesh symmetry points, *Numer. Math.*, **95** (2003), 781–801.

[29] Z. Zhang and A. Naga, A new finite element gradient recovery method: superconvergence property, *SIAM J. Sci. Comput.*, **26** (2005), 1192–1213.

[30] Q.D. Zhu, Higher Accuracy Post-processing Theory for Finite Element Methods (In Chinese). Science Publisher, Beijing, 2008.

[31] Q.D. Zhu and Q. Lin, The Superconvergence Theory of the Finite Element Methods (In Chinese), Hunan Science Publisher, Changsha, 1989.

[32] O.C. Zienkiewicz and J.Z. Zhu, A simple error estimator and adaptive procedure for practical engineering analysis, *Internat. J. Numer. Methods Engrg.*, **24** (1987), 337–357.

[33] O.C. Zienkiewicz and J.Z. Zhu, The superconvergence patch recovery and a posteriori error estimates, part I: the recovery technique, *Internat. J. Numer. Methods Engrg.*, **33** (1992), 1331–1364.