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SPECTRAL METHOD FOR MIXED INHOMOGENEOUS BOUNDARY VALUE PROBLEMS IN THREE DIMENSIONS*

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Abstract

In this paper, we investigate spectral method for mixed inhomogeneous boundary value problems in three dimensions. Some results on the three-dimensional Legendre approximation in Jacobi weighted Sobolev space are established, which improve and generalize the existing results, and play an important role in numerical solutions of partial differential equations. We also develop a lifting technique, with which we could handle mixed inhomogeneous boundary conditions easily. As examples of applications, spectral schemes are provided for three model problems with mixed inhomogeneous boundary conditions. The spectral accuracy in space of proposed algorithms is proved. Efficient implementations are presented. Numerical results demonstrate their high accuracy, and confirm the theoretical analysis well.

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Key words: Three-dimensional Legendre approximation in Jacobi weighted Sobolev space, Lifting technique, Spectral method for mixed inhomogeneous boundary value problems.

1. Introduction

The spectral method has gained increasing popularity in scientific computations, see [3, 4, 7, 8, 10, 11, 13] and the references therein. Recently, some authors developed the Jacobi spectral approximation, and enlarged the applications of spectral method, see [2, 14, 16–18]. There have been a lot of work on one (or two) -dimensional problems. However, it is also interesting to consider spectral method in three dimensions, cf. [1, 8, 9, 12, 20, 22].

In this paper, we investigate the Legendre spectral method for mixed inhomogeneous boundary value problems in three-dimensional space. In the next section, we first recall some recent results on the one-dimensional Legendre orthogonal approximation presented in [17]. By using those results, together with the interpolation of operators (cf. [6]), we establish the basic results on the three-dimensional Legendre orthogonal approximation in Jacobi weighted Sobolev space. The new results improve and generalize the existing results of [3,4,14,17], and play an important

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role in spectral method for partial differential equations with mixed boundary conditions. In particular, the existence of Jacobi weights in the norms appearing in the error estimates covers certain singularities of approximated functions. In Section 3, we consider spectral method for mixed inhomogeneous boundary value problems. We could handle such problems in two ways. The first way is to approximate the Dirichlet boundary conditions suitably, and then use certain proper approximations, see e.g., [1, 15] and the references therein. The second way is to reform the original problems to the related homogeneous boundary value problems, and then solve the resulting ones easily. For instance, the work of [19, 21] for Dirichlet boundary value problems and the work of [19] for mixed boundary value problems in two-dimensions. But, it is not easy to generalize this approach to three-dimensional problems. We also refer to the books [4,8,9,20]. We now develop an explicit lifting technique, with which we reformulate three model problems (steady or unsteady) with Dirichlet or mixed boundary conditions to some alternative forms with homogeneous Dirichlet boundary conditions imposed on some parts of the boundary. Then we provide the corresponding spectral schemes and prove their spectral accuracy. In Section 4, we describe the efficient numerical implementations, and present some numerical results demonstrating their high accuracy. The final section is for concluding remarks.

2. Legendre Orthogonal Approximation in Three Dimensions

In this section, we establish the new results on the Legendre orthogonal approximation in three dimensions.

2.1. One-dimensional Legendre orthogonal approximation

We first recall one-dimensional Legendre orthogonal approximation. Let $I = \{x \mid |x| < 1\}$. For $r \geq 0$, we define the Sobolev space $H^r(I)$ and its norm $\|u\|_{r,I}$ as usual. In particular, $L^2(I) = H^0(I)$ with the inner product $(u,v)_I$ and the norm $\|u\|_I$. Let $\mathcal{D}(I)$ be the set consisting of all infinitely differentiable functions with compact supports in I. $H_0^r(I)$ is the closure of $\mathcal{D}(I)$ in $H^r(I)$. For simplicity, we denote $\dfrac{\partial u}{\partial x}$ by $\partial_x u$, etc..

The Legendre polynomial of degree l is given by

$$L_l(x) = \frac{(-1)^l}{2^l l!}\partial_x^l (1-x^2)^l.$$

The set of all Legendre polynomials is a complete $L^2(I)$-orthogonal system.

For positive integer N, $\mathcal{P}_N(I)$ stands for the set of all polynomials of degree at most N. Throughout this paper, we denote by c a generic positive constant which does not depend on N and any function.

The orthogonal projection $P_{N,I}$: $L^2(I) \to \mathcal{P}_N(I)$ is defined by

$$(P_{N,I}u - u, \phi)_I = 0, \qquad \forall \phi \in \mathcal{P}_N(I).$$

By Theorem 2.1 of [17] with $\alpha = \beta = 0$, we know that if $u \in L^2(I), (1-x^2)^{\frac{s}{2}}\partial_x^s u \in L^2(I)$ and integers $0 \leq s \leq N+1$, then

$$||P_{N,I}u - u||_I \leq cN^{-s}||(1-x^2)^{\frac{s}{2}}\partial_x^s u||_I. \tag{2.1}$$

The orthogonal projection $P_{N,I}^1 : H^1(I) \to \mathcal{P}_N(I)$ is defined by

$$(\partial_x(P_{N,I}^1 u - u), \partial_x \phi)_I + (P_{N,I}^1 u - u, \phi)_I = 0, \qquad \forall\ \phi \in \mathcal{P}_N(I).$$