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CONVERGENCE ANALYSIS FOR SPECTRAL APPROXIMATION TO A SCALAR TRANSPORT EQUATION WITH A RANDOM WAVE SPEED*

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Abstract

This paper is concerned with the initial-boundary value problems of scalar transport equations with uncertain transport velocities. It was demonstrated in our earlier works that regularity of the exact solutions in the random spaces (or the parametric spaces) can be determined by the given data. In turn, these regularity results are crucial to convergence analysis for high order numerical methods. In this work, we will prove the spectral convergence of the stochastic Galerkin and collocation methods under some regularity results or assumptions. As our primary goal is to investigate the errors introduced by discretizations in the random space, the errors for solving the corresponding deterministic problems will be neglected.

Mathematics subject classification: 52B10, 65D18, 68U05.
Key words: Scalar transport equations, Analytic regularity, Stochastic Galerkin, Stochastic collocation, Spectral convergence.

1. Introduction

In numerical simulation, accounting for uncertainties in input quantities (such as model parameters, initial and boundary conditions, and geometry) becomes an important issue in recent years, especially in risk analysis, safety, and optimal design, see, e.g., [1, 7, 9, 20, 23, 27]. Many works have been recently devoted to the analysis and the implementation of the Stochastic Galerkin (SG) methods and Stochastic Collocation (SC) techniques for such problems. These methods are promising since they can exploit the possible regularity of the solution with respect to the stochastic parameters to achieve faster convergence. SG methods and SC methods can be classified as parametric techniques, since both approximate u, the solution of the underlying problems as a linear combination of suitable deterministic basis functions in probability space. The Stochastic Galerkin is a projection technique over a set of orthogonal polynomials with respect to the probability measure at hand [25, 26] and this methods is also called the general Polynomial Chaos (gPC) methods which is first introduced in [24], while Stochastic Collocation is a sum of Lagrangian interpolants over the probability space (see e.g., [10, 12, 17, 18]).

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Many numerical analysis results for the linear stochastic elliptic equation have been given by researchers. Babuška et al. [3, 4] analyze the convergence properties for both the SG methods and SC methods for the stochastic elliptic equation, they show that both two methods achieves exponential convergence provided that the input random data are infinitely differentiable with respect to the random variables, under very mild assumptions on the growth of such derivatives, as is the case for standard expansions of random fields. Schwab and co-workers [14,15] provided similar results for the stochastic parabolic problems and the second order wave equations with random coefficients. They also discussed the convergence properties of the *Best N-term* approximation. The application of stochastic spectral methods to hyperbolic problems of conservation laws poses additional challenges. Very few works have been investigated for uncertain hyperbolic problems, especially for theoretical part. The scalar wave equation with a random wave number has been treated with gPC methods by Gottlieb and Xiu [13]. After that, Tang and Zhou [22, 28] give some rigorous regularity results for the similar model problem, and show that the regularity results are important for the analysis of convergence rate of the SG methods and SC methods. In this paper, for the initial-boundary value problems of linear transport equation, we will show the analytic regularity of the solution with respect to the random parameter. Such results are crucial for analyzing the convergence properties of high order numerical methods. By using the analytic regularity results together with complex analysis, the spectral convergence of the Stochastic Galerkin and Collocation methods are shown. We note that related works on the second order wave equations with random data has been done by Nobile et.al. [6]. We also remark that numerical treatment for nonlinear hyperbolic problems are also discussed by many researchers, see, e.g., [19].

The paper is organized as follows. In Section 2, we set up the problems and discuss some analytic regularity results of the solutions in the parametric spaces. Spectral convergence of the Stochastic Gelerkin and collocation methods will be investigated in Section 3. Then, we provided with an numerical example in Section 4. Some conclusion remarks will be provided in the final section.

2. Problem Set Up and Solution Regularity

2.1. Problem set up

Let $x \in D \equiv [-1, 1]$ be the spatial coordinate, and t be the time variable in $T \equiv [0, T]$, and $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, whose event(ω) space is Ω and is equipped with σ-algebra $\mathcal{A}$, and $P : \mathcal{A} \to [0, 1]$ is a probability measure. We consider the following class of linear scalar transport equations with random velocity: Find a random function, $u : T \times D \times \Omega \to \mathbb{R}$ such that P-almost everywhere in Ω, or in other words, almost surely the following equation holds:

$$\frac{\partial u(x,t,y(\omega))}{\partial t} = c(y(\omega))\frac{\partial u(x,t,y(\omega))}{\partial x}, \tag{2.1a}$$

$$u(x,t=0,y(\omega)) = u_0(x,y(\omega)). \tag{2.1b}$$

A well-posed boundary conditions can be given by

$$u(-1,t;y(\omega)) = u_L(t;y(\omega)) \qquad c(y(\omega)) < 0, \tag{2.2a}$$

$$u(+1,t;y(\omega)) = u_R(t;y(\omega)) \qquad c(y(\omega)) > 0. \tag{2.2b}$$