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A DISCONTINUOUS GALERKIN METHOD FOR THE FOURTH-ORDER CURL PROBLEM*

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Abstract

In this paper, we present a discontinuous Galerkin (DG) method based on the Nédélec finite element space for solving a fourth-order curl equation arising from a magnetohydrodynamics model on a 3-dimensional bounded Lipschitz polyhedron. We show that the method has an optimal error estimate for a model problem involving a fourth-order curl operator. Furthermore, some numerical results in 2 dimensions are presented to verify the theoretical results.

Mathematics subject classification: 65N30.
Key words: Fourth-order curl problem, DG method, Nédélec finite element space, Error estimate.

1. Introduction

Magnetohydrodynamics (MHD) equations describe the macroscopic dynamics of electrical fluid that moves in a magnetic field. The MHD model is governed by Navier-Stokes equations coupled with Maxwell equations through Ohm's law and the Lorentz force. As an example, a resistive MHD system is described by the following equations:

$$\begin{cases} \rho(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \mu\triangle\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{B}_t - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \\ \qquad = -\frac{\eta}{\mu_0}\nabla\times(\nabla\times\boldsymbol{B}) - \frac{d_i}{\mu_0}\nabla\times((\nabla\times\boldsymbol{B})\times\boldsymbol{B}) - \frac{\eta_2}{\mu_0}(\nabla\times)^4\boldsymbol{B}, \\ \nabla\cdot\boldsymbol{B} = 0, \end{cases} \tag{1.1}$$

where ρ is the mass density, $\boldsymbol{u}$ is the velocity, p is the pressure, $\boldsymbol{B}$ is the magnetic induction field, η is the resistivity, η_2 is the hyper-resistivity, μ_0 is the magnetic permeability of free space,

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and μ is the viscosity. The primary variables in MHD equations are the fluid velocity $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$.

MHD models have wide applications in thermonuclear fusion, plasma physics, geophysics, and astrophysics. The mathematical modeling and numerical simulation of MHD have been the subject of considerable research effort in the past few decades, such that various numerical algorithms have been proposed in MHD simulations. In order to solve MHD equations that contain a fourth-order term, we will focus on using a DG method.

DG methods are effective methods in solving partial differential equations. As far back as 1973, Reed and Hill [33] proposed a DG method for the hyperbolic equation. Since then, DG methods have been used widely to solve hyperbolic problems (see, e.g., [14–17]) and elliptic problems (see, e.g., [1, 2, 10–12, 19, 35]). At the same time, they have important applications to other problems such as Navier-Stokes equations (see,e.g., [8, 18]), Euler equation [38], and fractional diffusion problem [34]. The use of DG methods for elliptic problems can be traced back to the penalty method [28] and the interior penalty (IP) method [6]. A lot of work (see, e.g., [2, 4–7, 28, 35]) has been done on DG methods for second-order elliptic equations. For fourth-order elliptic problems, Baker [7] used an IP method to solve the biharmonic problem on 2-dimensional smooth domains. Engel et al. [20] combined concepts from the continuous Galerkin method, the DG method and stabilization techniques to approximate fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. In addition, Brenner and Sung [9] used an IP method to solve the biharmonic problem on 2-dimensional bounded polygonal domains. Xu and Shu [37] reviewed the works on local DG methods for high-order time-dependent problems.

Recently, DG methods have also been applied in the numerical simulation of Maxwell equations. In 2002, Perugia et al. [32] used an IP method for the time harmonic Maxwell equation. In 2004, Cockburn et al. [13] used a local divergence-free DG method for the Maxwell equation. In their study, the approximate solution is preserved divergence-free on each element, and computational costs are much lower than for standard DG methods. Moreover, Houston et al. [22] used a mixed DG method to approximate the Maxwell operator; Lu et al. [29] gave a DG method for the Maxwell equation with Debye-type dissipative material and artificial PML (perfectly matched layer) boundary. In 2005, Houston et al. applied an IP method [23] and a mixed DG method [24] for the indefinite time harmonic Maxwell equation. Recently, Li [26, 27] considered an interior penalty DG method for the time-dependent Maxwell equations in cold plasma.

In the numerical simulation of the MHD equations (1.1), it is necessary to design an efficient numerical discretization for a fourth-order curl problem. As is well-known, constructing a curl-curl-conforming element for the fourth-order curl problem is very difficult. Zheng, Hu, and Xu [39] used a nonconforming finite element method to solve fourth-order curl equations. Motivated by the IP method for the fourth-order elliptic problem [9], in this paper, we design a DG method for solving the fourth-order curl problem. The main feature of this scheme is that we can use the standard higher-order Nédélec finite element space.

In this paper, we begin by introducing the fourth-order curl model equation. According to the model problem, we establish the corresponding variational problem by introducing suitable function spaces. By showing that the trial function space is a Hilbert space, we give the well-posedness of the variational problem and a regularity result of the weak solution. Second, based on the standard higher-order Nédélec finite element space, we design a DG method for the fourth-order curl problem and prove the boundedness and coercivity of the discrete variational problem. Finally, we prove the optimal error estimate and present some numerical results to verify the theoretical results.

The remainder of this paper is organized as follows: In Section 2, we introduce the model equation, establish the corresponding variational problem and give the well-posedness of the variational problem as well as the regularity of the weak solution. In Section 3, we propose a DG method for the fourth-order curl problem, prove the boundedness and coercivity of the discrete variational problem and obtain the optimal error estimate. In Section 4, we present some numerical results, and in Section 5 we offer some concluding remarks.

For convenience, we introduce the following notations [36].

Here $C_i, 1 \leq i \leq 4$, are positive constants and do not depend on x or y, we denote $x \leqslant C_1 y$ by $x \lesssim y$; $x \geqslant C_2 y$ by $x \gtrsim y$; and $C_3 y \leq x \leqslant C_4 y$ by $x \cong y$. $(\nabla\times\nabla\times\nabla\times\nabla\times) = (\nabla\times)^4$, $(\nabla\times\nabla\times\nabla\times) = (\nabla\times)^3$.

2. A Fourth-order Curl Problem

In this section, we introduce a model problem for the fourth-order magnetic induction equations above. By introducing appropriate test and trial function spaces, we propose the corresponding variational problem and show its well-posedness. Furthermore, we give a regularity result for the weak solution.

Through time discretization of the MHD equations given above and by ignoring the nonlinear terms, we obtain a partial differential equation containing a fourth-order curl operator. In order to derive a method for the numerical simulation of the equation, we consider a simplified model as follows:

$$\begin{cases} (\nabla\times)^4 \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f} \quad & \boldsymbol{x} \in \Omega, \\ \boldsymbol{u} \times \boldsymbol{n}|_{\partial\Omega} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{n}|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz polygonal domain, $\boldsymbol{f} \in (L^2(\Omega))^3$ and $div\boldsymbol{f} = 0$. We refer to this simplified model as the fourth-order curl problem. The choice of boundary conditions in (2.1) arises naturally in the variational formulation given in (2.5).

To study the fourth-order curl problem (2.1), we define the space of 3-dimensional vector functions with curl curl in L^2 by

$$V = \Big\{ \boldsymbol{u} \in (L^2(\Omega))^3 \mid \nabla\times\nabla\times\boldsymbol{u} \in (L^2(\Omega))^3 \Big\}.$$

We define the following bilinear form of space V,

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\nabla\times\nabla\times\boldsymbol{u}, \nabla\times\nabla\times\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the inner product of $(L^2(\Omega))^3$. We can easily prove that the bilinear form defined by (2.2) is an inner product of space V.

The inner product defined by (2.2) defines a norm $\|\cdot\|_V$ on space V as follows:

$$\|\boldsymbol{u}\|_V = \sqrt{(\boldsymbol{u}, \boldsymbol{u})_V}\ . \tag{2.3}$$

Lemma 2.1. *Let $\boldsymbol{u}$ be in $(C_0^\infty(\Omega))^3$, then the following estimate holds:*

$$\|\nabla\times\boldsymbol{u}\|_{(L^2(\Omega))^3} \leq \frac{1}{2}\Big(\|\nabla\times\nabla\times\boldsymbol{u}\|_{(L^2(\Omega))^3} + \|\boldsymbol{u}\|_{(L^2(\Omega))^3}\Big). \tag{2.4}$$

Proof. Integrating by parts and noting that $\boldsymbol{u}$ vanishes on the boundary, we have

$$\begin{aligned}\int_\Omega \nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{u}dx &= \int_\Omega \boldsymbol{u}\cdot\nabla\times\nabla\times\boldsymbol{u}dx + \int_{\partial\Omega}\boldsymbol{n}\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{u}ds\\ &= \int_\Omega \boldsymbol{u}\cdot\nabla\times\nabla\times\boldsymbol{u}dx.\end{aligned}$$

Using the Cauchy-Schwarz inequality in the above equation, we have

$$\|\nabla\times\boldsymbol{u}\|^2_{(L^2(\Omega))^3} \le \|\nabla\times\nabla\times\boldsymbol{u}\|_{(L^2(\Omega))^3}\|\boldsymbol{u}\|_{(L^2(\Omega))^3},$$

which implies that

$$\|\nabla\times\boldsymbol{u}\|_{(L^2(\Omega))^3} \le \frac{1}{2}\Big(\|\nabla\times\nabla\times\boldsymbol{u}\|_{(L^2(\Omega))^3} + \|\boldsymbol{u}\|_{(L^2(\Omega))^3}\Big).$$

This completes the proof. □

Let V_0 denote the closure of $(C_0^\infty(\Omega))^3$ in V. Define the space of 3-dimensional vector functions with curl in L^2 by

$$H_0(curl,\Omega) = \Big\{\boldsymbol{u}\in(L^2(\Omega))^3 \mid \nabla\times\boldsymbol{u}\in(L^2(\Omega))^3,\ \boldsymbol{u}\times\boldsymbol{n}=0\Big\}.$$

Corresponding to the definition of higher-order scalar Sobolev spaces, it is also convenient to define, for $k\ge 0$,

$$H^k(curl,\Omega) = \{\boldsymbol{u}\in(H^k(\Omega))^3 | \nabla\times\boldsymbol{u}\in(H^k(\Omega))^3\}.$$

By Lemma 2.1, we obtain that V_0 is a subspace of $H_0(curl,\Omega)$ and that V_0 can be characterized as

$$V_0 = \Big\{\boldsymbol{u}\in H_0(curl,\Omega) \mid \nabla\times\nabla\times\boldsymbol{u}\in(L^2(\Omega))^3,\ (\nabla\times\boldsymbol{u})\times\boldsymbol{n}|_{\partial\Omega}=0\ \Big\}.$$

Based on the Hilbert space V_0, we can easily obtain the equivalent variational problem of the fourth-order curl problem (2.1): Given $\boldsymbol{f}\in(L^2(\Omega))^3$ with $div\boldsymbol{f}=0$, find $\boldsymbol{u}\in V_0$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = \langle\boldsymbol{f},\boldsymbol{v}\rangle,\quad \forall\boldsymbol{v}\in V_0, \tag{2.5}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\times\nabla\times\boldsymbol{u}\cdot\nabla\times\nabla\times\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{v}dx. \tag{2.6}$$

By the Riesz representation theorem, we have the following theorem on the well-posedness of the variational problem (2.5).

Theorem 2.1. *There exists a unique solution $\boldsymbol{u}\in V_0$ of the variational problem* (2.5) *that satisfies*

$$\|\boldsymbol{u}\|_V \le \|\boldsymbol{f}\|_{(L^2(\Omega))^3}. \tag{2.7}$$

Next, we give a result on the regularity of the weak solution.

Theorem 2.2. *Let $\boldsymbol{u}$ be the weak solution of the fourth-order problem* (2.1)*, $\boldsymbol{f}\in(L^2(\Omega))^3$, $div\boldsymbol{f}=0$. Then there exists a $\delta\in(0,\frac{1}{2}]$ such that*

$$\boldsymbol{u},\ \nabla\times\boldsymbol{u}\in(H^{\frac{1}{2}+\delta}(\Omega))^3,$$

and such that the following estimates hold

$$\|\boldsymbol{u}\|_{(H^{1/2+\delta}(\Omega))^3} \lesssim \|\boldsymbol{f}\|_{(L^2(\Omega))^3},\quad \|\nabla\times\boldsymbol{u}\|_{(H^{1/2+\delta}(\Omega))^3} \lesssim \|\boldsymbol{f}\|_{(L^2(\Omega))^3}. \tag{2.8}$$

As it is well-known [21] that $H_0(curl,\Omega)\cap H(div,\Omega)\hookrightarrow(H^{1/2+\delta}(\Omega))^3$, the proof is obvious.

3. A DG Method for the Fourth-order Curl Problem

In this section, motivated by the IP method [9] for the fourth-order elliptic problem, we design a DG method for the fourth-order curl problem and prove an optimal error estimate.

3.1. Finite Element Space

Let $T_h = \{K\}$ be a shape-regular tetrahedron partition of Ω. We introduce the follwing Nédélec div-conforming and curl-conforming finite element spaces [31]:

$$V_h = \{\boldsymbol{u}_h \in H(div;\Omega) \mid \boldsymbol{u}_h|_K \in (P_k)^3,\ \forall\ K \in T_h\}, \tag{3.1}$$

$$X_h = \{\boldsymbol{u}_h \in H_0(curl;\Omega) \mid \boldsymbol{u}_h|_K \in (P_k)^3,\ \forall\ K \in T_h,\ k \geq 2\}. \tag{3.2}$$

Theorem 3.1. *Let Π_h^{div} be the canonical interpolation operator associated with the finite element space V_h, and let h be the diameter of a tetrahedron K. For every $\boldsymbol{p}$ in $(H^{k+1}(K))^3$, we have* [31]

$$\begin{aligned}
&\|\boldsymbol{p} - \Pi_h^{div}\boldsymbol{p}\|_{(L^2(K))^3} \lesssim h^{k+1}|\boldsymbol{p}|_{(H^{k+1}(K))^3},\\
&|\boldsymbol{p} - \Pi_h^{div}\boldsymbol{p}|_{(H^s(K))^3} \lesssim h^{k+1-s}|\boldsymbol{p}|_{(H^{k+1}(K))^3}, \quad 1 \leq s \leq k+1.
\end{aligned}$$

Theorem 3.2. *Let Π_h^{curl} be the canonical interpolation operator associated with the finite element space X_h, and let h be the diameter of a tetrahedron K. For every $\boldsymbol{p}$ in $(H^{k+1}(K))^3,\ k \geq 1$, we have* [31]

$$\begin{aligned}
&\|\boldsymbol{p} - \Pi_h^{curl}\boldsymbol{p}\|_{(L^2(K))^3} \lesssim h^{k+1}|\boldsymbol{p}|_{(H^{k+1}(K))^3},\\
&\Pi_h^{div}(\nabla \times \boldsymbol{p}) = \nabla \times (\Pi_h^{curl}\boldsymbol{p}).
\end{aligned}$$

3.2. Discrete Variational Problem

In this subsection we derive the discrete variational problem of the fourth-order curl problem (2.1).

Let $\boldsymbol{u}$ be the weak solution of the fourth-order curl problem (2.1), and assume that $\boldsymbol{u} \in H^2(curl,\Omega)$. For any $K \in T_h$, by multiplying $\boldsymbol{v}_h \in X_h$ on the both sides of equation (2.1) and by using Green's formula, we find

$$\begin{aligned}
\int_K \boldsymbol{f}\cdot\boldsymbol{v}_h\,dx = &\int_{\partial K} (\nabla\times)^3\boldsymbol{u}\cdot\boldsymbol{v}_h \times \boldsymbol{n}ds + \int_K \nabla\times\nabla\times\boldsymbol{u}\cdot\nabla\times\nabla\times\boldsymbol{v}_h\,dx\\
&+ \int_K \boldsymbol{u}\cdot\boldsymbol{v}_h\,dx + \int_{\partial K} \nabla\times\nabla\times\boldsymbol{u}\cdot(\nabla\times\boldsymbol{v}_h)\times\boldsymbol{n}ds.
\end{aligned} \tag{3.3}$$

Let $\mathcal{E}_h^0$ be the set of internal faces of partition T_h , $f \in \mathcal{E}_h^0$ be the interface of two adjacent elements $K^{\pm}$, and $\boldsymbol{n}^{\pm}$ be the unit outward normal vector of the face f associated with $K^{\pm}$ (Fig. 3.1). We denote by $\boldsymbol{v}^{\pm} = (\boldsymbol{v}|_{K^{\pm}})|_f$ and introduce two notations as follows:

$$[\![\nabla\times\boldsymbol{v}]\!] = (\nabla\times\boldsymbol{v}^+)\times\boldsymbol{n}^+ + (\nabla\times\boldsymbol{v}^-)\times\boldsymbol{n}^-, \tag{3.4}$$

$$\{\nabla\times\nabla\times\boldsymbol{v}\} = \frac{\nabla\times\nabla\times\boldsymbol{v}^+ + \nabla\times\nabla\times\boldsymbol{v}^-}{2}. \tag{3.5}$$

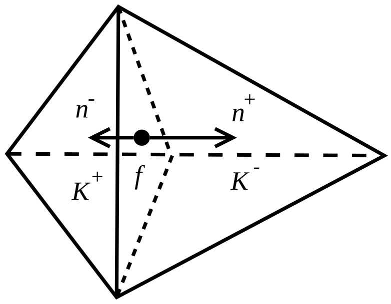

Fig. 3.1. f is the interface of two adjacent elements $K^{\pm}$.

Remark 3.1. If $f \in \mathcal{E}_h^\partial$, the set of boundary faces, we define $[\![\nabla\times\boldsymbol{v}]\!]$ and $\{\nabla\times\nabla\times\boldsymbol{v}\}$ on f, respectively, as follows:

$$[\![\nabla\times\boldsymbol{v}]\!] = (\nabla\times\boldsymbol{v})\times\boldsymbol{n}, \qquad \{\nabla\times\nabla\times\boldsymbol{v}\} = \nabla\times\nabla\times\boldsymbol{v}.$$

Take summation over all elements of T_h on both sides of (3.3), and notice that $\boldsymbol{v}_h\times\boldsymbol{n}$ is continuous over f. By notation (3.4), we have

$$a_h(\boldsymbol{u},\boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\ dx, \tag{3.6}$$

where

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}_h) = &\int_\Omega \nabla\times\nabla\times\boldsymbol{u}\cdot\nabla_h\times\nabla_h\times\boldsymbol{v}_h\ dx \\ &+\sum_{f\in\mathcal{E}_h}\int_f \nabla\times\nabla\times\boldsymbol{u}\cdot[\![\nabla\times\boldsymbol{v}_h]\!]\ ds + \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}_h\ dx, \end{aligned}$$

where $\nabla_h\times$ denotes the elementwise curl.

As $[\![\nabla\times\boldsymbol{u}]\!]=0$, noting that $a_h(\boldsymbol{u},\boldsymbol{v}_h)$ is not symmetric, using notation (3.5), we add the following zero term to $a_h(\boldsymbol{u},\boldsymbol{v}_h)$:

$$\sum_{f\in\mathcal{E}_h}\int_f \{\nabla\times\nabla\times\boldsymbol{v}_h\}\cdot[\![\nabla\times\boldsymbol{u}]\!]ds.$$

Then $a_h(\boldsymbol{u},\boldsymbol{v}_h)$ becomes a bilinear form satisfying symmetry. Furthermore, in order for $a_h(\boldsymbol{u},\boldsymbol{v}_h)$ to also satisfy coercivity with respect to the norm defined in (3.10), we add another zero term to $a_h(\boldsymbol{u},\boldsymbol{v}_h)$ as follows:

$$\sum_{f\in\mathcal{E}_h}\frac{\eta}{|e|}\int_f [\![\nabla\times\boldsymbol{u}]\!]\cdot[\![\nabla\times\boldsymbol{v}_h]\!]ds,$$

where $|e|$ denotes the diameter of the circumcircle of the face f, and where η denotes the penalty parameter. The above term is called a penalty term. Therefore, we have

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}_h) = &\int_\Omega \nabla_h\times\nabla_h\times\boldsymbol{u}\cdot\nabla_h\times\nabla_h\times\boldsymbol{v}_h dx \\ &+\sum_{f\in\mathcal{E}_h}\int_f \left(\{\nabla\times\nabla\times\boldsymbol{u}\}\cdot[\![\nabla\times\boldsymbol{v}_h]\!] + \{\nabla\times\nabla\times\boldsymbol{v}_h\}\cdot[\![\nabla\times\boldsymbol{u}]\!]\right) ds \\ &+\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}_h dx\ +\sum_{f\in\mathcal{E}_h}\frac{\eta}{|e|}\int_f [\![\nabla\times\boldsymbol{u}]\!]\cdot[\![\nabla\times\boldsymbol{v}_h]\!]ds. \end{aligned} \tag{3.7}$$

Thus, (3.6) is equivalent to

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \tag{3.8}$$

where $(\boldsymbol{f}, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h dx$.

Hence, we obtain the discrete variational problem of the fourth-order curl problem (2.1): Given $\boldsymbol{f} \in (L^2(\Omega))^3$ with $div\boldsymbol{f} = 0$, find $\boldsymbol{u}_h \in X_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \ \boldsymbol{v}_h \in \ X_h, \tag{3.9}$$

where $a_h(\cdot,\cdot)$ is defined by (3.7).

3.3. Well-posedness of the Discrete Variational Problem

Suppose $W = \{\boldsymbol{u} \mid \nabla\times\nabla\times\boldsymbol{u}|_K \in (H^{1/2+s}(K))^3, \ \forall\, K \in T_h\}$, where s is a positive constant. Let $X = X_h + V_0 \cap W$. We introduce a norm on X associated with the grid T_h as follows: For any $\boldsymbol{w} \in X$, we define

$$\begin{aligned}\|\boldsymbol{w}\|_h =&\Big(\sum_{K\in T_h} \|\nabla\times\nabla\times\boldsymbol{w}\|^2_{(L^2(K))^3} + \sum_{K\in T_h} \|\boldsymbol{w}\|^2_{(L^2(K))^3} \\ &+ \sum_{f\in\mathcal{E}_h} |e| \ \|\{\nabla\times\nabla\times\boldsymbol{w}\}\|^2_{(L^2(f))^3} + \sum_{f\in\mathcal{E}_h} |e|^{-1} \ \| [\![\nabla\times\boldsymbol{w}]\!]\|^2_{(L^2(f))^3}\Big)^{\frac{1}{2}},\end{aligned} \tag{3.10}$$

where $|e|$ denotes the diameter of the circumcircle of the face f, $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$. We can easily prove that (3.10) is a norm on X.

Theorem 3.3. *The bilinear form $a_h(\cdot,\cdot)$ is bounded with respect to the norm $\|\cdot\|_h$, i.e.,*

$$|a_h(\boldsymbol{w}, \boldsymbol{v})| \le (\eta+1) \ \|\boldsymbol{w}\|_h\|\boldsymbol{v}\|_h \qquad \forall\ \boldsymbol{w},\ \boldsymbol{v} \in X. \tag{3.11}$$

Theorem 3.4. *For sufficiently large η, we have*

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \ge C_s \ \|\boldsymbol{v}_h\|_h^2 \quad \forall\ \boldsymbol{v}_h \in X_h, \tag{3.12}$$

where C_s is a positive constant depending only on the shape-regularity of the partition T_h, the degree of the polynomial k, and the penalty parameter η.

Proof. By the Cauchy–Schwarz inequality, we find

$$\begin{aligned}&\Big|\sum_{f\in\mathcal{E}_h} \int_f \{\nabla\times\nabla\times\boldsymbol{v}_h\}\cdot[\![\nabla\times\boldsymbol{v}_h]\!]ds\Big| \\ \le& \frac{1}{2}\rho^{-1}\sum_{f\in\mathcal{E}_h} |e| \int_f |\{\nabla\times\nabla\times\boldsymbol{v}_h\}|^2 ds + \frac{1}{2}\rho \sum_{f\in\mathcal{E}_h} |e|^{-1}\int_f |[\![\nabla\times\boldsymbol{v}_h]\!]|^2 ds,\end{aligned} \tag{3.13}$$

where ρ is an arbitrary positive parameter.

Suppose $f \subset K^+ \cap K^-$. Noting that $\boldsymbol{v}_h \in X_h$, by (3.5) and the trace theorem, we have

$$\begin{aligned}&|e|\int_f |\{\nabla\times\nabla\times\boldsymbol{v}_h\}|^2 ds \\ \le& \left(|e|\int_{f|_{K^+}} |\nabla\times\nabla\times\boldsymbol{v}_h|^2 ds + |e|\int_{f|_{K^-}} |\nabla\times\nabla\times\boldsymbol{v}_h|^2 ds\right) \\ \le& C_1 \int_{K^+\cup K^-} |\nabla\times\nabla\times\boldsymbol{v}_h|^2 dx,\end{aligned} \tag{3.14}$$

where C_1 is a positive constant depending on the shape-regularity of the partition T_h.

Therefore, if $\boldsymbol{v}_h \in X_h$, by (3.10) and (3.14), we have

$$\begin{aligned}\frac{1}{2C_1+1}\|\boldsymbol{v}_h\|_h^2 \le \sum_{K\in T_h} &\left(\|\nabla\times\nabla\times \boldsymbol{v}_h\|^2_{(L^2(K))^3} + \|\boldsymbol{v}_h\|^2_{(L^2(K))^3}\right)\\ &+ \sum_{f\in\mathcal{E}_h} |e|^{-1}\|[\![\nabla\times \boldsymbol{v}_h]\!]\|^2_{(L^2(f))^3}.\end{aligned} \tag{3.15}$$

Using (3.7) and (3.13), we find

$$\begin{aligned}a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge &\sum_{K\in T_h}\left(\|\nabla\times\nabla\times \boldsymbol{v}_h\|^2_{(L^2(K))^3} + \|\boldsymbol{v}_h\|^2_{(L^2(K))^3}\right)\\ &+ \sum_{f\in\mathcal{E}_h}\eta|e|^{-1}\|\,[\![\nabla\times \boldsymbol{v}_h]\!]\|^2_{(L^2(f))^3}\\ &- \rho^{-1}\sum_{f\in\mathcal{E}_h}|e|\int_f |\{\nabla\times\nabla\times \boldsymbol{v}_h\}|^2 ds - \rho\sum_{f\in\mathcal{E}_h}|e|^{-1}\int_f |[\![\nabla\times \boldsymbol{v}_h]\!]|^2 ds.\end{aligned}$$

By (3.14), we have

$$\begin{aligned}a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge &\sum_{K\in T_h}\left(\|\nabla\times\nabla\times \boldsymbol{v}_h\|^2_{(L^2(K))^3} + \|\boldsymbol{v}_h\|^2_{(L^2(K))^3}\right)\\ &+ \sum_{f\in\mathcal{E}_h}\eta|e|^{-1}\|\,[\![\nabla\times \boldsymbol{v}_h]\!]\|^2_{(L^2(f))^3}\\ &- 2\rho^{-1}C_1\sum_{K\in T_h}\|\nabla\times\nabla\times \boldsymbol{v}_h\|^2_{(L^2(K))^3} - \rho\sum_{f\in\mathcal{E}_h}|e|^{-1}\|[\![\nabla\times \boldsymbol{v}_h]\!]\|^2_{(L^2(f))^3}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge &(1-2\rho^{-1}C_1)\sum_{K\in T_h}\left(\|\nabla\times\nabla\times \boldsymbol{v}_h\|^2_{(L^2(K))^3} + \|\boldsymbol{v}_h\|^2_{(L^2(K))^3}\right)\\ &+ (\eta-\rho)\sum_{f\in\mathcal{E}_h}|e|^{-1}\|\,[\![\nabla\times \boldsymbol{v}_h]\!]\|^2_{(L^2(f))^3}.\end{aligned}$$

If we choose ρ such that $1-2\rho^{-1}C_1>0$ and η such that $\eta-\rho>0$, then by using (3.15), we obtain

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge C_s\ \|\boldsymbol{v}_h\|_h^2,$$

where C_s is a positive constant depending on the shape-regularity of the partition T_h, the degree of the polynomial k, and the penalty parameter η. □

Remark 3.2. Usually, we can choose $\rho = 4C_1, \eta = \rho + \frac{1}{2}$ and thus $C_s = \frac{1}{4C_1+2}$.

3.4. Convergence Analysis

Theorem 3.5. *Let $\boldsymbol{u}$ be the weak solution of the fourth-order curl problem* (2.1) *and $\boldsymbol{u} \in H^2(curl,\Omega)$. Suppose $\boldsymbol{u}_h$ is the solution of the discrete variational problem* (3.9), *then the following estimate holds:*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \lesssim \inf_{\boldsymbol{v}_h\in X_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_h.$$

Proof. We first note, by (3.6) and (3.9), that

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h)=0 \quad \forall \;\; \boldsymbol{v}_h\in X_h.$$

For any $\boldsymbol{v}_h \in X_h$, by the coercivity (3.12), we have

$$\begin{aligned}\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h^2 &\lesssim a_h(\boldsymbol{v}_h-\boldsymbol{u}_h,\boldsymbol{v}_h-\boldsymbol{u}_h)\\ &= a_h(\boldsymbol{v}_h-\boldsymbol{u}_h+\boldsymbol{u}-\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u}_h)\\ &= a_h(\boldsymbol{v}_h-\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u}_h)+a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h-\boldsymbol{u}_h).\end{aligned}$$

Therefore, we get

$$\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h^2 \lesssim a_h(\boldsymbol{v}_h-\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u}_h).$$

By the boundedness (3.11), we obtain

$$\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h^2 \lesssim a_h(\boldsymbol{v}_h-\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u}_h)\lesssim \|\boldsymbol{u}-\boldsymbol{v}_h\|_h\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h.$$

Canceling $\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h$ from the two sides, we have

$$\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h \lesssim \|\boldsymbol{u}-\boldsymbol{v}_h\|_h.$$

Thus, by the triangle inequality we find

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \leqslant \|\boldsymbol{u}-\boldsymbol{v}_h\|_h+\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_h \lesssim \|\boldsymbol{u}-\boldsymbol{v}_h\|_h.$$

By the arbitrariness of v_h, we have

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \lesssim \inf_{\boldsymbol{v}_h\in X_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_h.$$

Hence, we complete the proof. □

Theorem 3.6. *Let* $\boldsymbol{u}$ *be the weak solution of the fourth-order curl problem* (2.1), $\boldsymbol{u}\in H^k(curl,\Omega), k\geq 2$*, and* T_h *be quasi-uniform. Then there exists the following error estimate:*

$$\|\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u}\|_h \lesssim h^{k-1}|\nabla\times\boldsymbol{u}|_{(H^k(\Omega))^3}.$$

Proof. By (3.10), we find

$$\begin{aligned}&\|\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u}\|_h\\ =&\Big(\sum_{K\in T_h}\|\nabla\times\nabla\times(\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u})\|^2_{(L^2(K))^3}+\sum_{K\in T_h}\|\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u}\|^2_{(L^2(K))^3}\\ &+\sum_{f\in\mathcal{E}_h}|e|\;\|\{\nabla\times\nabla\times(\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u})\}\|^2_{(L^2(f))^3}+\sum_{f\in\mathcal{E}_h}|e|^{-1}\;\|\,[\![\nabla\times(\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u})]\!]\|^2_{(L^2(f))^3}\Big)^{\frac12}\\ =&(I_1+I_2+I_3+I_4)^{\frac12}.\end{aligned}$$

We first estimate I_1. Using Theorem 3.2 and Theorem 3.1, we get

$$\begin{aligned}&\int_K|\nabla\times\nabla\times(\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u})|^2dx\\ =&\int_K|\nabla\times(\nabla\times\boldsymbol{u}-\Pi_h^{div}(\nabla\times\boldsymbol{u}))|^2dx\\ \leq&|\nabla\times\boldsymbol{u}-\Pi_h^{div}(\nabla\times\boldsymbol{u})|^2_{(H^1(K))^3}\lesssim h^{2k-2}|\nabla\times\boldsymbol{u}|^2_{(H^k(K))^3}.\end{aligned}\tag{3.16}$$

Using Theorem 3.2, the estimate of I_2 is obvious. Next, we estimate I_3. Let $f = K_1 \cap K_2$. By the trace inequality, Theorems 3.2 and 3.1, we obtain

$$\begin{aligned}
&|e| \int_f |\{\nabla \times \nabla \times (\boldsymbol{u} - \Pi_h^{curl} \boldsymbol{u})\}|^2 ds \\
&\lesssim |e| \Big(h^{-1} \|\nabla \times \nabla \times (\boldsymbol{u} - \Pi_h^{curl} \boldsymbol{u})\|^2_{(L^2(K_1 \cup K_2))^3} + h |\nabla \times \nabla \times (\boldsymbol{u} - \Pi_h^{curl} \boldsymbol{u})|^2_{(H^1(K_1 \cup K_2))^3} \Big) \\
&\lesssim |\nabla \times \boldsymbol{u} - \Pi_h^{div} (\nabla \times \boldsymbol{u})|^2_{(H^1(K_1 \cup K_2))^3} + h^2 |\nabla \times (\nabla \times \boldsymbol{u} - \Pi_h^{div} (\nabla \times \boldsymbol{u}))|^2_{(H^1(K_1 \cup K_2))^3} \\
&\lesssim h^{2k-2} |\nabla \times \boldsymbol{u}|^2_{(H^k(K_1 \cup K_2))^3} + h^2 |\nabla \times \boldsymbol{u} - \Pi_h^{div} (\nabla \times \boldsymbol{u})|^2_{(H^2(K_1 \cup K_2))^3} \\
&\lesssim h^{2k-2} |\nabla \times \boldsymbol{u}|^2_{(H^k(K_1 \cup K_2))^3}. \qquad (3.17)
\end{aligned}$$

Finally, we estimate I_4. Also, let $f = K_1 \cap K_2$. By the trace inequality, Theorems 3.2 and 3.1, we find

$$\begin{aligned}
&|e|^{-1} \int_f |[\![\nabla \times (\boldsymbol{u} - \Pi_h^{curl} \boldsymbol{u})]\!]|^2 ds \\
&\lesssim |e|^{-1} \left(h^{-1} \|\nabla \times (\boldsymbol{u} - \Pi_h^{curl} \boldsymbol{u}) \|^2_{(L^2(K_1 \cup K_2))^3} + h |\nabla \times (\boldsymbol{u} - \Pi_h^{curl} \boldsymbol{u})|^2_{(H^1(K_1 \cup K_2))^3} \right) \\
&\lesssim h^{-2} \|\nabla \times \boldsymbol{u} - \Pi_h^{div} (\nabla \times \boldsymbol{u})\|^2_{(L^2(K_1 \cup K_2))^3} + |\nabla \times \boldsymbol{u} - \Pi_h^{div} (\nabla \times \boldsymbol{u}))|^2_{(H^1(K_1 \cup K_2))^3} \\
&\lesssim h^{2k-2} |\nabla \times \boldsymbol{u}|^2_{(H^k(K_1 \cup K_2))^3}. \qquad (3.18)
\end{aligned}$$

From the estimates of I_1, I_2, I_3, and I_4, we complete the proof. □

Based on Theorems 3.5 and 3.6, we obtain the following optimal error estimate.

Theorem 3.7. *Let $\boldsymbol{u}$ be the weak solution of the fourth-order curl problem* (2.1)*, and let $\boldsymbol{u} \in H^k(curl, \Omega)$, $k \geq 2$, and T_h be quasi-uniform. Suppose $\boldsymbol{u}_h$ is the solution of the discrete variational problem* (3.9)*, then the following estimate holds:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \lesssim h^{k-1} |\nabla \times \boldsymbol{u}|_{(H^k(\Omega))^3}.$$

4. Numerical Results

In this section, we present some numerical results that verify Theorem 3.6 in 2 dimensions. Then by Theorem 3.5, Theorem 3.7 is also verified.

Let $\Omega \subset R^2$ be a connected bounded domain, T_h be a uniform partition of Ω, $\mathcal{E}_h^0$ be the set of internal edges of partition T_h , $e \in \mathcal{E}_h^0$ be the interface of two adjacent elements $K^{\pm}$, and $\boldsymbol{\tau}$ be the unit tangential vector of the edge e. We denote by $\boldsymbol{v}^{\pm} = (\boldsymbol{v}|_{K^{\pm}})|_e$ and introduce two notations as follows

$$\begin{aligned}
[\![\nabla \times \boldsymbol{v}]\!] &= \nabla \times \boldsymbol{v}^+ - \nabla \times \boldsymbol{v}^-, \\
\{\nabla \times \nabla \times \boldsymbol{v}\} &= \frac{\nabla \times \nabla \times \boldsymbol{v}^+ \cdot \boldsymbol{\tau} + \nabla \times \nabla \times \boldsymbol{v}^- \cdot \boldsymbol{\tau}}{2},
\end{aligned}$$

which are a little different from the definitions of jump and mean in 3 dimensions. Furthermore, we use the second family of Nédélec quadratic elements, namely,

$$X_h = \Big\{ \boldsymbol{u}_h \in H_0(curl; \Omega) \mid \boldsymbol{u}_h|_K \in (P_k)^3, \ \forall\, K \in T_h, \ k = 2 \Big\}.$$

Let K be an element, the vertices of K are denoted by a_i, a_j, a_k, and the barycenter functions corresponding to a_i, a_j, a_k are denoted by $\lambda_i, \lambda_j, \lambda_k$.

The edge basis functions are defined thus: Let $[a_i, a_j]$ be an edge of the element, such that the three basis functions on the edge are

$$\boldsymbol{\varphi}_{ij}^1 = \lambda_i^2 \nabla\lambda_j, \quad \boldsymbol{\varphi}_{ij}^2 = \lambda_j^2 \nabla\lambda_i, \quad \boldsymbol{\varphi}_{ij}^3 = \lambda_i\lambda_j \nabla(\lambda_j - \lambda_i),$$

and the corresponding dual basis functions are

$$q_{ij}^1 = 9\lambda_i^2 - 18\lambda_i\lambda_j + 3\lambda_j^2, \quad q_{ij}^2 = -3\lambda_i^2 + 18\lambda_i\lambda_j - 9\lambda_j^2, \quad q_{ij}^3 = -9\lambda_i^2 + 42\lambda_i\lambda_j - 9\lambda_j^2.$$

The element basis functions are defined as follows: Let $[a_i, a_j, a_k]$ be the element, such that the three basis functions are

$$\boldsymbol{\psi}_{ijk}^1 = \lambda_i\lambda_j \nabla\lambda_k, \quad \boldsymbol{\psi}_{ijk}^2 = \lambda_i\lambda_k \nabla\lambda_j, \quad \boldsymbol{\psi}_{ijk}^3 = \lambda_j\lambda_k \nabla\lambda_i,$$

and the corresponding dual basis functions are

$$\begin{aligned}
\boldsymbol{q}_{ijk}^1 &= (-6 + 30\lambda_i)\boldsymbol{\tau}_{ik} + (-6 + 30\lambda_j)\boldsymbol{\tau}_{jk}, \\
\boldsymbol{q}_{ijk}^2 &= (-6 + 30\lambda_i)\boldsymbol{\tau}_{ik} + (-18 + 30\lambda_j)\boldsymbol{\tau}_{jk}, \\
\boldsymbol{q}_{ijk}^3 &= (-18 + 30\lambda_i)\boldsymbol{\tau}_{ik} + (-6 + 30\lambda_j)\boldsymbol{\tau}_{jk},
\end{aligned}$$

where $\boldsymbol{\tau}_{ik} = a_k - a_i = \overrightarrow{a_i a_k}, \boldsymbol{\tau}_{jk} = a_k - a_j = \overrightarrow{a_j a_k}$.

There are three edges on an element, so that the total number of basis functions is $3\times3+3 = 12$. The interpolation of $\boldsymbol{u}$ on an element K can be written as

$$(\Pi_h^{curl}\boldsymbol{u})|_K = \sum_{n=1}^{3}(u_{ij}^n\boldsymbol{\varphi}_{ij}^n + u_{jk}^n\boldsymbol{\varphi}_{jk}^n + u_{ik}^n\boldsymbol{\varphi}_{ik}^n) + \sum_{m=1}^{3} u_{ijk}^m\boldsymbol{\psi}_{ijk}^m,$$

where

$$\begin{aligned}
u_{ij}^n &= \frac{1}{|e_{ij}|}\int_{e_{ij}} \boldsymbol{u}\cdot\boldsymbol{\tau}_{ij} q_{ij}^n ds, \quad u_{jk}^n = \frac{1}{|e_{jk}|}\int_{e_{jk}} \boldsymbol{u}\cdot\boldsymbol{\tau}_{jk} q_{jk}^n ds, \\
u_{ik}^n &= \frac{1}{|e_{ik}|}\int_{e_{ik}} \boldsymbol{u}\cdot\boldsymbol{\tau}_{ik} q_{ik}^n ds,
\end{aligned}$$

similar to $\boldsymbol{\tau}_{ik}$, here $\boldsymbol{\tau}_{ij}$ defined as $a_j - a_i = \overrightarrow{a_i a_j}$, and

$$\begin{aligned}
u_{ijk}^m = \frac{1}{|K|}\{\int_K \boldsymbol{u}\cdot\boldsymbol{q}_{ijk}^m dx - \sum_{n=1}^{3}(u_{ij}^n\int_K \boldsymbol{\varphi}_{ij}^n\cdot\boldsymbol{q}_{ijk}^m dx + u_{jk}^n\int_K \boldsymbol{\varphi}_{jk}^n\cdot\boldsymbol{q}_{ijk}^m dx \\
+ u_{ik}^n\int_K \boldsymbol{\varphi}_{ik}^n\cdot\boldsymbol{q}_{ijk}^m dx)\}.
\end{aligned}$$

We now set $\Omega = [0,1]^2$, $\boldsymbol{u} = (3\pi\sin^2\pi y\cos\pi y\sin^3\pi x, -3\pi\sin^2\pi x\cos\pi x\sin^3\pi y)$. In order to verify the theoretical result in Theorem 3.6, we need to compute the relative error:

$$\text{relative error} = \frac{\|\boldsymbol{u}-\Pi_h^{curl}\boldsymbol{u}\|_h}{|\nabla\times\boldsymbol{u}|_{(H^2(\Omega))^3}}.$$

From Table 1, we see that the numerical results confirm the theoretical result in Theorems 3.6 and 3.7 can also be verified from Theorem 3.5.

Table 4.1: The convergence of the relative error.

$h=\frac{1}{N}$	relative error	order
$N=4$	8.2872×10^{-2}	–
$N=8$	4.0325×10^{-2}	1
$N=16$	1.9727×10^{-2}	1
$N=32$	9.7826×10^{-3}	1
$N=64$	4.8770×10^{-3}	1
$N=128$	2.4358×10^{-3}	1
$N=256$	1.2173×10^{-3}	1
$N=512$	6.0825×10^{-4}	1
$N=1024$	3.0424×10^{-4}	1

5. Conclusion

In this paper, we discussed a DG method for a fourth-order curl problem. We first established the related variational problem, and then gave the well-posedness and regularity of the weak solution.

Based on the Nédélec finite element space, we designed a DG method for the fourth-order curl problem, proposed the discrete variational problem of the method, and proved the boundedness and coercivity of the discrete variational problem. We thereby obtained an optimal error estimate. Some numerical results in 2 dimensions were also presented in this paper.

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