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A POSTERIORI ERROR ESTIMATE OF FINITE ELEMENT METHOD FOR THE OPTIMAL CONTROL WITH THE STATIONARY BÉNARD PROBLEM*

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Abstract

In this paper, we consider the adaptive finite element approximation for the distributed optimal control associated with the stationary Bénard problem under the pointwise control constraint. The states and co-states are approximated by polynomial functions of lowest-order mixed finite element space or piecewise linear functions and control is approximated by piecewise constant functions. We give the a posteriori error estimates for the control, the states and co-states.

Mathematics subject classification: 49J20, 65N30.
Key words: Optimal control problem, Stationary Bénard problem, Nonlinear coupled system, A posteriori error estimate.

1. Introduction

The control of viscous flow for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The Boussinesq approximation of the Navier-Stokes system is frequently used as mathematical model for fluid flow in semiconductor melts. In many crystal growth technics, such as Czochralski growth and zone-melting technics, the behavior of the flow has considerable impact on the crystal quality. It is therefore quite natural to establish flow conditions that guarantee desired crystal properties. As control actions, they include distributed forcing, distributed heating, and others. For example, the control of vorticity has significant applications in science and engineering such as the control of turbulence and control of crystal growth process.

Considerable progress has been made in mathematics physics and computation for the optimal control problems of the viscous flow; see [1, 2, 9, 11, 12] and reference therein. Optimal control problems of the thermally coupled incompressible Navier-Stokes equation by Neumann and Dirichlet boundary heat controls were considered in [11, 12]. Also, the time dependent problems were considered in the literature. In this article, we consider the Bénard problem whose state is governed by the Boussinesq equations, which are crucial to many technological and scientific applications. Without the control constraint, the approximation for the optimal control of the stationary Bénard problem was considered in [16], and it used the gradient iterative method to solve the discretized equations. For the constrained control case, there seems to

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be little work on this problem. This paper is concerned with the finite element approximation for the constrained optimal control problem of the stationary Bénard problem:

$$(\mathcal{P})\quad \min_{Q\in K}\quad J(Q)=\{\frac{1}{2}\|\mathbf{u}-\mathbf{U}\|^2_{\mathbf{L}^2}+\frac{\alpha}{2}\|Q\|^2_{0,\Omega}\},$$

subject to the Boussinesq system:

$$\begin{aligned}
&(a)\quad -\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=T\mathbf{g}+f\quad in\ \Omega,\\
&(b)\quad \nabla\cdot\mathbf{u}=0\quad in\ \Omega,\\
&(b)\quad -\kappa\Delta T+\mathbf{u}\cdot\nabla T=Q\quad in\ \Omega,\\
&(c)\quad \mathbf{u}=0\quad T=0\quad on\ \partial\Omega
\end{aligned}\tag{1.1}$$

and subject to the control constraint

$$K=\Big\{Q\in L^2(\Omega):Q(x)\geq d>0;\ a.e.\ x\in\Omega\Big\},\tag{1.2}$$

where Ω is the regular bounded and convex open set in $\mathbb{R}^n$ $(n=2,\ or\ 3)$, with $\partial\Omega\in C^{1,1}$. $\mathbf{u},p,T$ denote the velocity, pressure and temperature fields, respectively, f is a body force, and the control Q. The vector $\mathbf{g}$ is in the direction of gravitational acceleration and $\kappa>0$ is the thermal conductivity parameter. In this paper we only consider, for the simplicity, the case where κ is constant. Assume $\nu>0$ is the kinematic viscosity.

The optimal control problem $(\mathcal{P})$ is to seek the state variables $(\mathbf{u},p,T)$ and Q such that the functional J is minimized subject to (1.1) where $\mathbf{U}$ is some desired velocity fields. The physical target of the minimization problem is to match a desired flow field by adjusting the distributed control Q.

Adaptive finite element approximation is of very importance in improving accuracy and efficiency of the finite element discretisation. It ensures a higher density of nodes in certain area of the given domain, where the solution is more difficult to approximate, using a posteriori error indicator. In this sense, efficiency and reliability of adaptive finite element approximation rely very much on the error indicator used. Recently adaptive mesh refinement has been found quite useful in computing optimal control problem governed by elliptic equations, see [19], for example. Usually the control variable has only limited regularity. Thus suitable adaptive mesh can quite efficiently reduce the approximation error. There have been very extensive studies on the a posteriori error estimates and convergence analysis for the optimal control problems governed by elliptic or time dependent equations; see, for example, [22,24,25] and [19,23,26] and the references cited therein. However there seems to exist few known results on the a posteriori error estimates for the above control problem governed by the coupled nonlinear equations.

The paper is organized as follows. In Section 2, we give some notations and assumptions that will be used throughout the paper. In Section 3, we will discuss the finite element approximation for the optimal control problem. Section 4 contains the a posteriori error estimate for the optimal control problem in this article.

2. Notations and Preliminaries

Using the classical techniques, it can be proved that the optimal control problem has at least one solution. The reader is referred to [15, 18] for the details.