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A CHARACTERISTIC FINITE ELEMENT METHOD FOR CONSTRAINED CONVECTION-DIFFUSION-REACTION OPTIMAL CONTROL PROBLEMS*

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Abstract

In this paper, we develop a priori error estimates for the solution of constrained convection-diffusion-reaction optimal control problems using a characteristic finite element method. The cost functional of the optimal control problems consists of three parts: The first part is about integration of the state over the whole time interval, the second part refers to final-time state, and the third part is a regularization term about the control. We discretize the state and co-state by piecewise linear continuous functions, while the control is approximated by piecewise constant functions. Pointwise inequality function constraints on the control are considered, and optimal a L^2-norm priori error estimates are obtained. Finally, we give two numerical examples to validate the theoretical analysis.

Mathematics subject classification: 49J20, 65M15, 65M25, 65M60.
Key words: Characteristic finite element method, Constrained optimal control, Convection-diffusion-reaction equations, Pointwise inequality constraints, A priori error estimates.

1. Introduction

Optimal control problems governed by convection-diffusion equations arise in many scientific and engineering applications, such as atmospheric pollution control problems [1, 2]. Efficient numerical methods are essential to successful applications of such optimal control problems. To the best of the authors' knowledge, recently there are some growing published results on optimal control problems governed by steady convection-diffusion equations; see [3,4] of SUPG method, [5] of the standard finite element discretizations with stabilization, [6] of symmetric stabilization method, [7] of edge-stabilization method, [8] of the application of RT mixed DG scheme, [9] of domain decomposition method and so on. However, for the approximation of constrained optimal control problems governed by time-dependent convection-diffusion equations, it is much more complicated and only a few paper has been published, see [10–13] for example. Systematic

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introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [14–17].

In many time-dependent optimal control problems, people are usually interested in an optimization of the final-time state $y(x,T)$. Therefore, in this paper we consider the cost functional consisting of three parts: The first part is about integration of the state over the whole time interval, the second part refers to final-time state, and the third part is a regularization term about the control. Besides, we discuss the pointwise inequality function constraints on the control. In what follows, we shall study in details the following convection-diffusion-reaction state equations:

$$\begin{aligned}&\partial_t y(x,t) - \mu\Delta y(x,t) + \mathbf{a}(x)\cdot\nabla y(x,t) + c(x)y(x,t)\\ &= f(x,t) + u(x,t) \quad \text{in } \Omega\times(0,T],\end{aligned} \tag{1.1}$$

combined with the following boundary and initial conditions

$$y(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T], \quad y(x,0) = y_0(x) \quad \text{in } \Omega, \tag{1.2}$$

and

$$\alpha(x,t) \le u(x,t) \le \beta(x,t) \quad \text{a.e. in } \Omega\times[0,T], \tag{1.3}$$

where $\mathbf{a}$, c, f, α, β are given functions, $\mu > 0$ is a constant diffusion coefficient. Detailed assumptions for model problems will be introduced in Section 2.

It is well known that for the above convection-diffusion-reaction equations standard finite element discretization may not work. The methods of characteristics [19, 20] combine the convection and capacity terms in the governing equations, to carry out the temporal discretization in a Lagrange coordinate. These methods are symmetric and stable, even if large time steps and coarse spatial meshes are used. Thus, in this work we apply a characteristic finite element method to constrained optimal control problems governed by convection-diffusion-reaction equations. Pointwise inequality constraints on the control are also considered, and we obtain optimal a L^2-norm error estimates for both the control and state approximations.

The outline of the paper is as follows: In Section 2, we review the model problems and derive the continuous optimality conditions. In Section 3, we describe the characteristic finite element discretization of (1.1)-(1.3) and formulate a corresponding discretized optimality conditions. In Section 4, we prove a L^2-norm error estimates for the optimal control problems with control constraints. In Section 5, some numerical experiments are presented to observe the convergence behavior of the proposed numerical scheme.

In this paper, we denote C and δ be a generic constant and small positive number, which are independent of the discrete parameters and may have different values in different circumstances, respectively.

2. Model Problems and Optimality Conditions

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Just for simplicity of presentation, we assume that Ω is a convex polygon. Throughout this paper, we use the standard notations $L^p(\Omega)$ $(1\le p\le\infty)$ for Lebesgue space of real-valued functions with norm $\|\cdot\|_{L^p(\Omega)}$, and $W^{m,p}(\Omega)$ $(1\le p\le\infty)$ for Sobolev spaces endowed with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and semi-norm $|\cdot|_{W^{m,p}(\Omega)}$. For $p=2$, we denote $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$, $W^{m,2}(\Omega) = H^m(\Omega)$ and we drop the subscript $p=2$ in the corresponding norms and semi-norms. We also denote by