A NOTE ON JACOBI SPECTRAL-COLLOCATION METHODS FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS WITH SMOOTH SOLUTIONS*

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Abstract

This work is concerned with spectral Jacobi-collocation methods for Volterra integral equations of the second kind with a weakly singular of the form $(t - s)^{-\alpha}$. When the underlying solutions are sufficiently smooth, the convergence analysis was carried out in [Chen & Tang, J. Comput. Appl. Math., 233 (2009), pp. 938-950]; due to technical reasons the results are restricted to $0 < \mu < \frac{1}{2}$. In this work, we will improve the results to the general case $0 < \mu < 1$ and demonstrate that the numerical errors decay exponentially in the infinity and weighted norms when the smooth solution is involved.

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1. Introduction

We consider the linear Volterra integral equations (VIEs) of the second kind with weakly singular kernels

$$y(t) = g(t) + \int_0^t (t - s)^{-\mu} K(t,s)y(s)ds, \qquad t \in I, \tag{1.1}$$

where $I = [0, T]$, the function $g \in C(I)$, $y(t)$ is the unknown function, $\mu \in (0, 1)$ and $K \in C(I \times I)$ with $K(t,t) \neq 0$ for $t \in I$. Several numerical methods have been proposed for (1.1) (see, e.g., [1,5,17,18]). For (1.1) without the singular kernel (i.e., $\mu = 0$), spectral methods and the corresponding error analysis have been provided recently [19].

As the first derivatives of the solution $y(t)$ behave like $y'(t) \sim t^{-\mu}$ (see, e.g., [1]), it is difficult to employ high order numerical methods for solving (1.1). In [4], a Jacobi-collocation spectral method is developed for (1.1). To handle the non-smoothness of the underlying solutions, both function transformation and variable transformation are used to change the equation into a new Volterra integral equation defined on the standard interval $[-1, 1]$. However, the function transformation (see also [5]) generally makes the resulting equations and approximations more

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complicated. We also point out a relevant recent work [8] where we consider the case with $\mu = 1/2$ (the so-called Abel integral equations) and with *non-smooth* solutions. In [8], only coordinate transformation is used but the solution transformation is not used; where in [4] both transformations are employed.

On the other hand, due to the presence of the singular factor $(t - s)^{-\mu}$ in (1.1), even with the smoothness assumption of the underlying solutions there are still difficulties in establishing a framework to obtain spectral accuracy (i.e., errors decay exponentially with the increase the degree of freedom). The study of establishing the framework was carried out in [3], but due to the technical reason the convergence results were obtained for $0 < \mu < \frac{1}{2}$ only. The main purpose of this work is to extend the results in [3] to more general values of μ, i.e., $0 < \mu < 1$. The main technical difference between this work and [3] will be pointed out at the end of Section 2. Moreover, unlike [4,8], neither coordinate nor solution transformation will be used due to the smoothness assumption of the underlying solutions.

This paper is organized as follows. In Section 2, we outline the spectral approaches for (1.1). Some lemmas useful for establishing the convergence results will be provided in Section 3. The convergence analysis will be carried out in Section 4.

2. Jacobi-collocation Methods

Let $\omega^{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$ be a weight function in the usual sense, for $\alpha,\ \beta > -1$. As defined in [2,7,16], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ forms a complete $L_{\omega^{\alpha,\beta}}^2(-1,1)$-orthogonal system, where $L_{\omega^{\alpha,\beta}}^2(-1,1)$ is a weighted space defined by

$$L_{\omega^{\alpha,\beta}}^2(-1,1) = \{v : \ v \text{ is measurable and } ||v||_{\omega^{\alpha,\beta}} < \infty\},$$

equipped with the norm

$$||v||_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} |v(x)|^2 \omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

and the inner product

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx, \qquad \forall\ u,v \in L_{\omega^{\alpha,\beta}}^2(-1,1).$$

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N + 1)$ Jacobi Gauss points, corresponding to the weight $\omega^{-\mu,-\mu}(x)$. Let $\mathcal{P}_N$ denote the space of all polynomials of degree not exceeding N. For any $v \in C[-1,1]$, we can define the Lagrange interpolating polynomial $I_N^{\alpha,\beta}v \in \mathcal{P}_N$, satisfying

$$I_N^{\alpha,\beta}v(x_i) = v(x_i), \qquad 0 \le i \le N, \tag{2.1}$$

see, e.g., [2,7,16]. The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta}v(x) = \sum_{i=0}^{N} v(x_i)F_i(x),$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^N$.

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t = \frac{1}{2}T(1 + x), \qquad s = \frac{1}{2}T(1 + \tau),$$

and let

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad f(x) = g\left(\frac{T}{2}(1+x)\right). \tag{2.2}$$

Following [3], we obtain from (1.1) that

$$u(x) = f(x) + \int_{-1}^{x} (x-\tau)^{-\mu} \widetilde{K}(x,\tau) u(\tau) d\tau, \qquad x \in [-1,1], \tag{2.3a}$$

where

$$\widetilde{K}(x,\tau) = \left(\frac{T}{2}\right)^{1-\mu} K\left(\frac{T}{2}(1+x), \frac{T}{2}(1+\tau)\right). \tag{2.3b}$$

Firstly, Eq. (2.3a) holds at the collocation points $\{x_i\}_{i=0}^{N}$ on [-1,1], associated with $\omega^{-\mu,-\mu}$:

$$u(x_i) = f(x_i) + \int_{-1}^{x_i} (x_i-\tau)^{-\mu} \widetilde{K}(x_i,\tau) u(\tau) d\tau, \qquad 0 \le i \le N. \tag{2.4}$$

In order to obtain high order accuracy for the VIEs problem, the main difficulty is to compute the integral term in (2.4). We rewrite the integral term as:

$$\int_{-1}^{x_i} (x_i-\tau)^{-\mu} \widetilde{K}(x_i,\tau) u(\tau) d\tau = \int_{-1}^{1} (1-\theta)^{-\mu} K_1(x_i, \tau_i(\theta)) u(\tau_i(\theta)) d\theta, \tag{2.5}$$

by using the following variable change

$$\tau = \tau_i(\theta) = \frac{1+x_i}{2}\theta + \frac{x_i-1}{2}, \qquad \theta \in [-1,1], \tag{2.6}$$

where

$$K_1(x_i, \tau_i(\theta)) = \left(\frac{1+x_i}{2}\right)^{1-\mu} \widetilde{K}(x_i, \tau_i(\theta)). \tag{2.7}$$

Next, using a $(N+1)$-point Gauss quadrature formula relative to the Jacobi weight $\{w_i\}_{i=0}^{N}$, the integration term in (2.4) can be approximated by

$$\int_{-1}^{1} (1-\theta)^{-\mu} K_1(x_i, \tau_i(\theta)) u(\tau_i(\theta)) d\theta \sim \sum_{k=0}^{N} K_1(x_i, \tau_i(\theta_k)) u(\tau_i(\theta_k)) w_k, \tag{2.8}$$

where the set $\{\theta_k\}_{k=0}^{N}$ is the set of $(N+1)$ Jacobi Gauss, or Jacobi Gauss-Radau, or Jacobi Gauss-Lobatto points, corresponding to the weight $\omega^{-\mu,0}$. We use u_i, $0 \le i \le N$, to approximate the function value $u(x_i)$, $0 \le i \le N$, and use

$$u^N(x) = \sum_{j=0}^{N} u_j F_j(x) \tag{2.9}$$

to approximate the function $u(x)$, namely, $u(x_i) \approx u_i, u(x) \approx u^N(x)$, and

$$u(\tau_i(\theta_k)) \approx \sum_{j=0}^{N} u_j F_j(\tau_i(\theta_k)). \tag{2.10}$$

Then, the Jacobi collocation method is to seek $u^N(x)$ such that $\{u_i\}_{i=0}^{N}$ satisfies the following collocation equations:

$$u_i = f(x_i) + \sum_{j=0}^{N} u_j \left(\sum_{k=0}^{N} K_1(x_i, \tau_i(\theta_k)) F_j(\tau_i(\theta_k)) w_k\right), \quad 0 \le i \le N. \tag{2.11}$$

We close this section by pointing out the technical difference between this work and [3]. The main difference is the choice of $\{x_i\}$ in (2.4) and $\{\theta_k\}$ in (2.8). In this work, $\{x_i\}$ is associated with $\omega^{-\mu,-\mu}$ and $\{\theta_k\}$ is associated with $\omega^{-\mu,0}$, i.e., they are different; but in [3] both are associated with $\omega^{-\mu,0}$, i.e., $x_i = \theta_i$. Note our analysis requires to use the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials (for details, see Lemma 3.3). It will be demonstrated that if $\{x_i\}$ and $\{\theta_k\}$ are chosen differently, we can make better use of appropriate Lebesgue constants. This in turn will extend our analysis to the general values of μ.

3. Some Useful Lemmas

Throughout the paper C will denote a generic positive constant that is independent of N but which will depend on the length T of the interval $I = [0, T]$ and on bounds for the given functions f, $\widetilde{K}$ which is defined in (2.3b), and the index μ.

We first introduce some weighted Hilbert spaces. For simplicity, denote $\partial_x v(x) = (\partial/\partial x)v(x)$, etc. For non-negative integer m, define

$$H^m_{\omega^{\alpha,\beta}}(-1,1) := \left\{ v : \ \partial_x^k v \in L^2_{\omega^{\alpha,\beta}}(-1,1), \ 0 \leq k \leq m \right\},$$

with the semi-norm and the norm as

$$|v|_{m,\omega^{\alpha,\beta}} = ||\partial_x^m v||_{\omega^{\alpha,\beta}}, \qquad ||v||_{m,\omega^{\alpha,\beta}} = \left(\sum_{k=0}^{m} |v|^2_{k,\omega^{\alpha,\beta}} \right)^{1/2},$$

respectively.

To bound approximation errors of Jacobi polynomials, only some of the L^2-norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the semi-norms

$$|v|^{m;N}_{\omega^{\alpha,\beta}} := |v|_{H^{m;N}_{\omega^{\alpha,\beta}}(-1,1)} = \left(\sum_{k=\min(m,N+1)}^{m} ||\partial_x^k v||^2_{L^2_{\omega^{\alpha,\beta}}(-1,1)} \right)^{\frac{1}{2}}.$$

Lemma 3.1. ([2,15]) For any function v satisfying $v \in H^m_{\omega^{\alpha,\beta}}(-1,1)$, with $-1 < \alpha, \ \beta < 1$, we have

$$||v - I_N^{\alpha,\beta} v||_{\omega^{\alpha,\beta}} \leq CN^{-m}|v|^{m;N}_{\omega^{\alpha,\beta}}, \tag{3.1}$$

$$||v - I_N^{\alpha,\beta} v||_{1,\omega^{\alpha,\beta}} \leq CN^{1-m}|v|^{m;N}_{\omega^{\alpha,\beta}}, \tag{3.2}$$

for the three families of Jacobi Gauss points.

Let us define a discrete inner product. For any $u, v \in C[-1, 1]$, define

$$(u, v)_N = \sum_{j=0}^{N} u(x_j)v(x_j)w_j. \tag{3.3}$$

Lemma 3.2. ([2,15]) If $v \in H^m_{\omega^{\alpha,\beta}}(-1,1)$ for some $m \geq 1$ and $\phi \in \mathcal{P}_N$, then for the three families of Jacobi Gauss integration we have

$$|(v, \phi)_{\omega^{\alpha,\beta}} - (v, \phi)_N| \leq CN^{-m}|v|^{m;N}_{\omega^{\alpha,\beta}}||\phi||_{\omega^{\alpha,\beta}}. \tag{3.4}$$

From [10], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials.

Lemma 3.3. *Let $\{F_j(x)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss points of the Jacobi polynomials. Then*

$$||I_N^{\alpha,\beta}||_\infty := \max_{x\in[-1,1]} \sum_{j=0}^N |F_j(x)| = \begin{cases} \mathcal{O}\left(\log N\right), & -1 < \alpha, \ \beta \le -\frac{1}{2}, \\ \mathcal{O}\left(N^{\gamma+\frac{1}{2}}\right), & \gamma = \max(\alpha,\beta), \ \text{otherwise.} \end{cases} \tag{3.5}$$

For $r \ge 0$ and $\kappa \in [0,1]$, $C^{r,\kappa}([-1,1])$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm

$$||v||_{r,\kappa} = \max_{0\le k\le r} \max_{x\in[-1,1]} |\partial_x^k v(x)| + \max_{0\le k\le r} \sup_{x\ne y\in[-1,1]} \frac{|\partial_x^k v(x) - \partial_x^k v(y)|}{|x-y|^\kappa}.$$

When $\kappa = 0$, $C^{r,0}([-1,1])$ denotes the space of functions with r continuous derivatives on $[-1,1]$, which is also commonly denoted by $C^r([-1,1])$, and with norm $||\cdot||_r$.

We shall make use of a result of Ragozin [12,13], which states that, for non-negative integers r and $\kappa \in (0,1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\mathcal{T}_N v \in \mathcal{P}_N$ such that

$$||v - \mathcal{T}_N v||_\infty \le C_{r,\kappa} N^{-(r+\kappa)} ||v||_{r,\kappa}. \tag{3.6}$$

Actually, as stated in [12,13], $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}([-1,1])$ into $\mathcal{P}_N$.

We further define a linear, weakly singular integral operator $\mathcal{M}$:

$$\mathcal{M}v = \int_{-1}^x (x-\tau)^{-\mu} \widetilde{K}(x,\tau) v(\tau) d\tau. \tag{3.7}$$

Below we will show that $\mathcal{M}$ is compact as an operator from $C([-1,1])$ to $C^{0,\kappa}([-1,1])$ provided that the index κ satisfies $0 < \kappa < 1-\mu$. The proof of the following lemma can be found in [4].

Lemma 3.4. *Let $\kappa \in (0,1)$ and $\mathcal{M}$ be defined by (3.7). Then for any function $v \in C([-1,1])$, there exists a positive constant C, which is dependent of $||\widetilde{K}||_{0,\kappa}$, such that*

$$\frac{|\mathcal{M}v(x') - \mathcal{M}v(x'')|}{|x'-x''|^\kappa} \le C \max_{x\in[-1,1]} |v(x)|, \tag{3.8}$$

under the assumption that $0 < \kappa < 1-\mu$, for any $x', \ x'' \in [-1,1]$ and $x' \ne x''$. This implies that

$$||\mathcal{M}v||_{0,\kappa} \le C||v||_\infty, \qquad 0 < \kappa < 1-\mu, \tag{3.9}$$

where $||\cdot||_\infty$ is the standard norm in $C([-1,1])$.

4. Convergence Analysis

The objective of this section is to analyze the approximation scheme (2.11). Firstly, we derive the error estimate in L^∞ norm of the Jacobi collocation method.

4.1. Error estimate in L^∞

Theorem 4.1. *Let u be the exact solution to the Volterra integral equation (2.3) and the approximated solution u^N be obtained by using the spectral collocation scheme (2.11) together with a polynomial interpolation (2.9). If μ associated with the weakly singular kernel satisfies $0 < \mu < 1$ and $u \in H^m_{\omega^{-\mu,-\mu}}(-1,1)$ ($m \geq 1$), then*

$$||u - u^N||_\infty \leq \begin{cases} CN^{\frac{1}{2}-m}\left(|u|^{m;N}_{\omega^{-\mu,-\mu}} + N^{-\frac{1}{2}}\log N K^*||u||_\infty\right), & \frac{1}{2} \leq \mu < 1, \\ CN^{\frac{1}{2}-m}\left(|u|^{m;N}_{\omega^{-\mu,-\mu}} + N^{-\mu}K^*||u||_\infty\right), & 0 < \mu < \frac{1}{2}, \end{cases} \tag{4.1}$$

for N sufficiently large, where

$$K^* = \max_{0 \leq i \leq N} |K_1(x_i, \tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}. \tag{4.2}$$

Proof. First, we use the weighted inner product to rewrite (2.4) as

$$u(x_i) = f(x_i) + (K_1(x_i, \tau_i(\cdot)), u(\tau_i(\cdot)))_{\omega^{-\mu,0}}, \qquad 0 \leq i \leq N. \tag{4.3}$$

By using the discrete inner product (3.3), we set

$$(K_1(x_i, \tau_i(\cdot)), \phi(\tau_i(\cdot)))_N = \sum_{k=0}^N K_1(x_i, \tau_i(\theta_k))\phi(\tau_i(\theta_k))w_k.$$

Then, the numerical scheme (2.11) can be written as

$$u_i = f(x_i) + \left(K_1(x_i, \tau_i(\cdot)), u^N(\tau_i(\cdot))\right)_N, \qquad 0 \leq i \leq N, \tag{4.4}$$

where u^N is defined by (2.9). Subtracting (4.4) from (4.3) gives the error equation:

$$u(x_i) - u_i = (K_1(x_i, \tau_i(\cdot)), e(\tau_i(\cdot)))_{\omega^{-\mu,0}} + I_{i,2}$$
$$= \int_{-1}^{x_i} (x_i - \tau)^{-\mu}\widetilde{K}(x_i, \tau)e(\tau)d\tau + I_{i,2}, \tag{4.5}$$

for $0 \leq i \leq N$, where $e(x) = u(x) - u^N(x)$ is the error function,

$$I_{i,2} = \left(K_1(x_i, \tau_i(\cdot)), u^N(\tau_i(\cdot))\right)_{\omega^{-\mu,0}} - \left(K_1(x_i, \tau_i(\cdot)), u^N(\tau_i(\cdot))\right)_N,$$

and the integral transformation (2.5) was used here. Using the integration error estimates from Jacobi-Gauss polynomials quadrature in Lemma 3.2, we have

$$|I_{i,2}| \leq CN^{-m}|K_1(x_i, \tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}||u^N(\tau_i(\cdot))||_{\omega^{-\mu,0}}. \tag{4.6}$$

Multiplying $F_i(x)$ on both sides of the error equation (4.5) and summing up from $i = 0$ to $i = N$ yield

$$I_N^{-\mu,-\mu}u - u^N$$
$$= I_N^{-\mu,-\mu}\left(\int_{-1}^x (x - \tau)^{-\mu}\widetilde{K}(x, \tau)e(\tau)d\tau\right) + \sum_{i=0}^N I_{i,2}F_i(x). \tag{4.7}$$

Consequently,

$$e(x) = \int_{-1}^x (x - \tau)^{-\mu}\widetilde{K}(x, \tau)e(\tau)d\tau + I_1 + I_2 + I_3, \tag{4.8}$$

where

$$I_1 = u - I_N^{-\mu,-\mu}u, \qquad I_2 = \sum_{i=0}^{N} I_{i,2}F_i(x), \tag{4.9a}$$

$$I_3 = I_N^{-\mu,-\mu}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)d\tau\right)$$
$$- \int_{-1}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)d\tau. \tag{4.9b}$$

It follows from the Gronwall inequality (see [4])

$$||e||_\infty \le C\Big(||I_1||_\infty + ||I_2||_\infty + ||I_3||_\infty\Big). \tag{4.10}$$

From Sobolev inequality (see p.490, A.12, [2]) and Lemma 3.1, we obtained that

$$||I_1||_\infty = ||u - I_N^{-\mu,-\mu}u||_\infty$$
$$\le C||u - I_N^{-\mu,-\mu}u||^{1/2}_{\omega^{-\mu,-\mu}}||u - I_N^{-\mu,-\mu}u||^{1/2}_{1,\omega^{-\mu,-\mu}}$$
$$\le CN^{\frac{1}{2}-m}|u|^{m;N}_{\omega^{-\mu,-\mu}}. \tag{4.11}$$

Next, from (4.6), we have

$$\max_{0\le i\le N}|I_{i,2}| \le CN^{-m}\max_{0\le i\le N}|K_1(x_i,\tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}\max_{0\le i\le N}||u^N(\tau_i(\cdot))||_{\omega^{-\mu,0}}$$
$$\le CN^{-m}\max_{0\le i\le N}|K_1(x_i,\tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}||u^N||_\infty$$
$$\le CN^{-m}\max_{0\le i\le N}|K_1(x_i,\tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}\left(||e||_\infty + ||u||_\infty\right). \tag{4.12}$$

Hence, by using (4.9a), (4.12) and Lemma 3.3, we have

$$||I_2||_\infty = \left\|\sum_{i=0}^{N}I_{i,2}F_i(x)\right\|_\infty \tag{4.13}$$

$$\le C\max_{0\le i\le N}|I_{i,2}|\max_{x\in[-1,1]}\sum_{j=0}^{N}|F_j(x)|$$

$$\le \begin{cases} CN^{-m}\log N\max_{0\le i\le N}|K_1(x_i,\tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}\left(||e||_\infty + ||u||_\infty\right), & \frac{1}{2}\le\mu<1, \\ CN^{\frac{1}{2}-\mu-m}\max_{0\le i\le N}|K_1(x_i,\tau_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}\left(||e||_\infty + ||u||_\infty\right), & 0<\mu<\frac{1}{2}, \end{cases}$$

for sufficiently large N. We now estimate the third term I_3. Note that

$$I_N^{-\mu,-\mu}p(x) = p(x), \quad i.e., \quad (I_N^{-\mu,-\mu} - I)p(x) = 0, \qquad \forall\, p(x) \in \mathcal{P}_N. \tag{4.14}$$

As for the bound of $||I_3||_\infty$, we use the same idea as [4]. It follows from (4.14), (3.6), and Lemma 3.3 that

$$||I_3||_\infty = ||(I_N^{-\mu,-\mu} - I)\mathcal{M}e||_\infty \le \begin{cases} CN^{-\kappa}\log N||e||_\infty, & \frac{1}{2}\le\mu<1, \\ CN^{\frac{1}{2}-\mu-\kappa}||e||_\infty, & 0<\mu<\frac{1}{2}, \end{cases} \tag{4.15}$$

where in the last step we have used Lemma 3.4 under the following assumption:

$$\begin{cases} 0<\kappa<1-\mu, & \text{when } \frac{1}{2}\le\mu<1, \\ \frac{1}{2}-\mu<\kappa<1-\mu, & \text{when } 0<\mu<\frac{1}{2}. \end{cases}$$

It is clear that

$$||I_3||_\infty \le \frac{1}{3}||e||_\infty, \tag{4.16}$$

provided that N is sufficiently large. Combining (4.10), (4.11), (4.13) and (4.16) gives the desired estimate (4.1). □

4.2. Error estimate in weighted L^2 norm

To prove the error estimate in weighted L^2 norm, we need the generalized Hardy's inequality with weights (see, e.g., [9, 14]).

Lemma 4.1. *For all measurable function $f \geq 0$, the following generalized Hardy's inequality*

$$\left(\int_a^b |(Tf)(x)|^q u(x)dx\right)^{1/q} \leq C \left(\int_a^b |f(x)|^p v(x)dx\right)^{1/p}$$

holds if and only if

$$\sup_{a<x<b} \left(\int_x^b u(t)dt\right)^{1/q} \left(\int_a^x v^{1-p'}(t)dt\right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1}$$

for the case $1 < p \leq q < \infty$. Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x,t)f(t)dt$$

with $k(x,t)$ a given kernel, u, v weight functions, and $-\infty \leq a < b \leq \infty$.

From Theorem 1 in [11], we have the following weighted mean convergence result of Lagrange interpolation based at the zeros of Jacobi polynomials.

Lemma 4.2. *For every bounded function $v(x)$, there exists a constant C independent of v such that*

$$\sup_N \left\|\sum_{j=0}^N v(x_j)F_j(x)\right\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)} \leq C\|v\|_\infty,$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{x_i\}_{i=0}^N$.

Theorem 4.2. *Let u be the exact solution to the Volterra integral equation (2.3) and the approximated solution u^N be obtained by using the spectral collocation scheme (2.11) together with a polynomial interpolation (2.9). Assume that $u \in H^m_{\omega^{-\mu,-\mu}}(-1,1)$ $(m \geq 1)$, then, for N sufficiently large*

$$\begin{aligned}
&\|u - u^N\|_{\omega^{-\mu,-\mu}} \\
&\leq \begin{cases} CN^{-m}\left(N^{\frac{1}{2}-\kappa}|u|^{m;N}_{\omega^{-\mu,-\mu}} + K^*\|u\|_\infty\right), & \frac{1}{2} \leq \mu < 1, 0 < \kappa < 1-\mu, \\ CN^{-m}\left(|u|^{m;N}_{\omega^{-\mu,-\mu}} + K^*\|u\|_\infty\right), & 0 < \mu < \frac{1}{2}, \frac{1}{2} < \kappa < 1-\mu, \end{cases}
\end{aligned} \tag{4.17}$$

where K^* is defined by (4.2).

Proof. It follows from the Gronwall's lemma (see [4]) and (4.8) that

$$e(x) \leq C \int_{-1}^x (x-\tau)^{-\mu} \widetilde{K}(x,\tau)(I_1 + I_2 + I_3)(\tau)d\tau + I_1 + I_2 + I_3. \tag{4.18}$$

By the generalized Hardy's inequality Lemma 4.1, we obtain that

$$||e||_{\omega^{-\mu,-\mu}} \le C \left\| \int_{-1}^{x} (x-\tau)^{-\mu} \widetilde{K}(x,\tau)(I_1 + I_2 + I_3)(\tau)d\tau \right\|_{\omega^{-\mu,-\mu}}$$
$$+ C\Big(||I_1||_{\omega^{-\mu,-\mu}} + ||I_2||_{\omega^{-\mu,-\mu}} + ||I_3||_{\omega^{-\mu,-\mu}} \Big)$$
$$\le C\Big(||I_1||_{\omega^{-\mu,-\mu}} + ||I_2||_{\omega^{-\mu,-\mu}} + ||I_3||_{\omega^{-\mu,-\mu}} \Big). \tag{4.19}$$

Now, using Lemma 3.1, we obtain that

$$||I_1||_{\omega^{-\mu,-\mu}} = ||u - I_N^{-\mu,-\mu}u||_{\omega^{-\mu,-\mu}} \le CN^{-m}|u|_{\omega^{-\mu,-\mu}}^{m;N}. \tag{4.20}$$

Using Lemma 4.2 and (4.12) gives

$$||I_2||_{\omega^{-\mu,-\mu}} = \left\| \sum_{i=0}^{N} I_{i,2} F_i(x) \right\|_{\omega^{-\mu,-\mu}} \le C \max_{0 \le i \le N} |I_{i,2}|$$
$$\le CN^{-m} K^* \left(||e||_\infty + ||u||_\infty \right). \tag{4.21}$$

Finally, for the bound of $||I_3||_\infty$, we use the same idea as [4]. It follows from (4.14), Lemma 4.2, and (3.6) that

$$||I_3||_{\omega^{-\mu,-\mu}} = ||(I_N^{-\mu,-\mu} - I)\mathcal{M}e||_{\omega^{-\mu,-\mu}} \le CN^{-\kappa}||e||_\infty, \tag{4.22}$$

where in the last step we used Lemma 3.4 for any $\kappa \in (0, 1-\mu)$. By the convergence result in Theorem 4.1, we obtain that

$$||I_3||_{\omega^{-\mu,-\mu}} \le \begin{cases} CN^{\frac{1}{2}-m-\kappa} \left(|u|_{\omega^{-\mu,-\mu}}^{m;N} + N^{-\frac{1}{2}} \log N K^* ||u||_\infty \right), & \frac{1}{2} \le \mu < 1, \\ CN^{\frac{1}{2}-m-\kappa} \left(|u|_{\omega^{-\mu,-\mu}}^{m;N} + N^{-\mu} K^* ||u||_\infty \right), & 0 < \mu < \frac{1}{2}, \end{cases} \tag{4.23}$$

for N sufficiently large and for any $\kappa \in (0, 1-\mu)$. The desired estimate (4.17) is obtained by combining (4.20), (4.22), and (4.23). $\qquad \square$

5. Concluding Remarks

In this work, we have considered the Volterra integral equation (1.1) with the assumption that the exact solutions of (1.1) are smooth. We point out that this case may occur when the source function g in (1.1) is non-smooth; see, e.g., Theorem 6.1.11 in [1]. In this case, the Jacobi-collocation spectral method can be applied directly, which leads to spectral accuracy without using any coordinate or function transformations.

We close this work by noting that the Jacobi weighted Besov/Sobolev may be a natural tool for polynomial approximations. In [6, 7], it was demonstrated that Jacobi-weighted Besov and Sobolev spaces are the most appropriate tools for obtaining optimal upper and lower bounds when dealing with weakly singular problems, particularly for those with certain singularity at the end-points. This approach may be useful for analyzing the direct Jacobi-collocation spectral approach outlined in this work. It remains to be a future research topic.

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