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ON THE NONLINEAR MATRIX EQUATION $X^s + A^*F(X)A = Q$ with $s \geq 1$*

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Abstract

This work is concerned with the nonlinear matrix equation $X^s + A^*F(X)A = Q$ with $s \geq 1$. Several sufficient and necessary conditions for the existence and uniqueness of the Hermitian positive semidefinite solution are derived, and perturbation bounds are presented.

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Key words: Nonlinear matrix equations, Perturbation bound, Hermitian positive definite solution.

1. Introduction

Let $M(n)$ be the set of all $n \times n$ matrices and $P(n)$ be the set of all $n \times n$ Hermitian positive semidefinite matrices. We consider nonlinear matrix equation

$$X^s + A^*F(X)A = Q \ (s \geq 1), \tag{1.1}$$

where $A \in M(n)$, Q is an $n \times n$ Hermitian positive definite matrix, F is a map from $P(n)$ onto $P(n)$ or $-P(n)$, and the Hermitian positive semidefinite solution X is sought. Here A^* denotes the conjugate transpose of the matrix A. Note that X is a solution of (1.1) if and only if it is a fixed point of

$$G(X) = (Q - A^*F(X)A)^{\frac{1}{s}}.$$

The interest to study (1.1) arose, in particular, in connection with algebraic Riccati equations [2,6,18,21], interpolation [27,30] and the analysis of ladder networks, dynamic programming, control theory, stochastic filtering and statistics [2,16,17]. If $s = 1$, $F(X) = -X^{-1}$, the equation can be written in the form $X = Q + A^*X^{-1}A$. X is a solution of $X = Q + A^*X^{-1}A$ if and only if it is a solution of $X = Q + A^*(Q + A^*X^{-1}A)^{-1}A$. Assuming that A is invertible, this equation can be written as

$$X - F^*XF + F^*X(R + X)^{-1}XF - Q = 0,$$

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where $F = A^{-*}A$, $R = AQ^{-1}A^*$. This is a special case of the discrete algebraic Riccati equation

$$X - S^*XS + S^*XB(R + B^*XB)^{-1}B^*XS - Q = 0,$$

where $Q = Q^*$ and $R = R^*$ is invertible. For detail, see [21]. Several authors have considered such a nonlinear matrix equation, see [2,9-17,19-21,24-29,31-33,35,36] and [23]. It can be categorized as a general system of nonlinear equations in $\mathbb{C}^{n^2}$ space (see [3-7]), which includes the linear and nonlinear matrix equations recently discussed in [3-7,18,24] as special cases.

In [15], El-Sayed and Ran discussed a set of equations of the form $X + A^*F(X)A = Q$, where F maps positive definite matrices either onto positive definite matrices or onto negative definite matrices, and satisfies some monotonicity property. Ran and Reurings [26] also considered the equation $X + A^*F(X)A = Q$. They derived the solutions and perturbation theory. In [29], a perturbation analysis for nonlinear self-adjoint operator equations $X = Q \pm A^*F(X)A$ was provided. Based on the elegant properties of the Thompson metric, Liao, Yao and Duan [24] discussed the equation $X^s - A^*F(X)A = Q$ $(s > 1)$, where $F : P(n) \to P(n)$ is a self-adjoint and nonexpansive map.

The paper is organized as follows. In Section 2, we consider the case $F : P(n) \to P(n)$. The necessary and sufficient conditions for the existence of Hermitian positive semidefinite solution of the matrix equation are derived. A sufficient condition for the existence of a unique Hermitian positive semidefinite solution of the matrix equation is given. Finally, perturbation bounds between (1.1) and the perturbed equation

$$X^s + \widetilde{A}^*F(X)\widetilde{A} = \widetilde{Q} \ (s \geq 1) \tag{1.2}$$

are presented, where $\widetilde{A}$ and $\widetilde{Q}$ are small perturbations of A and Q, respectively. In Section 3, we discuss the case $F : P(n) \to -P(n)$ in a similar way as Section 2. Finally, in Section 4, we give some numerical examples.

Throughout this paper, we write $A \geq B (A > B)$ if both A and B are Hermitian and $A - B$ is positive semidefinite (definite). In particular, $A \geq 0 (A > 0)$ means that A is a Hermitian positive semidefinite (definite) matrix. $\varphi(n)$ denotes the closed set $\{X \in P(n) | X \geq Q^{\frac{1}{s}}\}$. Further, the sets $[A, B]$ and (A, B) are defined by $[A, B] = \{C | A \leq C \leq B\}$, $(A, B) = \{C | A < C < B\}$, whereas $L_{A,B}$ denotes the line segment joining A and B, i.e., $L_{A,B} = \{tA + (1-t)B | t \in [0, 1]\}$. See [28] for more details about these matrix orderings. We use $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ to denote the maximal and the minimal eigenvalues of an $n \times n$ Hermitian positive definite matrix X. $\| \cdot \|$, $\| \cdot \|_2$ and $\| \cdot \|_F$ denote the unitary invariant norm, the spectral norm and the Frobenius norm, respectively.

2. The Case $F : P(n) \to P(n)$

In this section, we derive some necessary and sufficient conditions for the existence and the uniqueness of a solution of (1.1) in the case that $F : P(n) \to P(n)$. The perturbation bound is presented.

Lemma 2.1. ([34]) *If $A \geq B \geq 0$ and $0 \leq r \leq 1$, then*

$$A^r \geq B^r. \tag{2.1}$$

Theorem 2.1. *Let $F : P(n) \to P(n)$ be continuous on $[0, Q^{\frac{1}{s}}]$.*