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# AN ACCELERATED WAVEFORM RELAXATION APPROACH BASED ON MODEL ORDER REDUCTION FOR LARGE COUPLING SYSTEMS*

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**Abstract**

In this paper, we present an accelerated simulation approach on waveform relaxation using Krylov subspace for a large time-dependent system composed of some subsystems. This approach first allows these subsystems to be decoupled by waveform relaxation. Then the Arnoldi procedure based on Krylov subspace is provided to accelerate the simulation of the decoupled subsystems independently. For the new approach, the convergent conditions on waveform relaxation are derived. The robust behavior is also successfully illustrated via numerical examples.

*Mathematics subject classification:* 78M34, 65Pxx.
*Key words:* Large coupling systems, Waveform relaxation, Model order reduction, Krylov subspace, Convergence analysis, Accelerating technique.

## 1. Introduction

Time-dependent systems are widely used to model and simulate complex physical processes. With the rapid development of the very large-scale integration technology, the dimension of time-dependent systems often becomes very large. For simulating such systems, it becomes extremely important to seek robust numerical simulation methods. Due to this fact, how to numerically solve large time-dependent systems has attracted extensive attention.

In this paper, we consider a time-dependent system composed of $k$ subsystems. For convenience sake, we assume that each subsystem is a linear time-invariant system described as

$$E_j \frac{dx_j(t)}{dt} = A_j x_j(t) + B_j u_j(t),\ y_j(t) = C_j x_j(t),\ j = 1, 2, \cdots, k, \tag{1.1}$$

with the initial conditions $x_j(t_0)$, and

$$\begin{cases} u_j(t) = F_{j1}y_1(t) + F_{j2}y_2(t) + \cdots + F_{jk}y_k(t) + G_j u(t), \\ y(t) = H_1 y_1(t) + H_2 y_2(t) + \cdots + H_k y_k(t), \end{cases} \tag{1.2}$$

where $E_j, A_j \in \mathbb{R}^{n_j \times n_j} (j = 1, 2, \cdots, k)$, $B_j \in \mathbb{R}^{n_j \times m_j}$, $C_j \in \mathbb{R}^{p_j \times n_j}$, $F_{ji} \in \mathbb{R}^{m_j \times p_i} (i = 1, 2, \cdots, k)$, $G_j \in \mathbb{R}^{m_j \times m}$, $H_j \in \mathbb{R}^{p \times p_j}$, $x_j(t) \in \mathbb{R}^{n_j}$ are internal state variables, $u_j(t) \in \mathbb{R}^{m_j}$ are internal inputs, $y_j(t) \in \mathbb{R}^{p_j}$ are internal outputs, $u(t) \in \mathbb{R}^m$ is an external input, and $y(t) \in \mathbb{R}^p$ is an external output. To our knowledge, this kind of system frequently arises in

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numerous research areas such as circuit simulation, control models, and discretizations of partial differential equations.

In the literature, waveform relaxation (WR), also known as dynamic iteration, is an effective technique to solve coupled systems described by ordinary differential equations and partial differential equations, for details see [4–6, 9, 10]. The WR technique allows coupled systems to be independently solved with its own time step length. Two typical WR schemes are the Jacobi and Gauss-Seidel relaxation processes. For simulating the system (1.1), the WR technique is an effective decoupling method. For example, for the $j$-th ($j = 1, 2, \cdots, k$) subsystem of (1.1), an iterative form of WR can be constructed as

$$\begin{aligned} & E_{1j}\frac{dx_j^{(l+1)}(t)}{dt} - \Big(A_{1j} + B_{1j}F_{1,jj}C_{1j}\Big)x_j^{(l+1)}(t) \\ =& E_{2j}\frac{dx_j^{(l)}(t)}{dt} - \Big(A_{2j} + B_{2j}F_{2,jj}C_{2j}\Big)x_j^{(l)}(t) + B_j\Big(\sum_{i=1,i\neq j}^{k} F_{ji}C_ix_i^{(l)}(t) + G_ju(t)\Big), \end{aligned} \tag{1.3}$$

where $E_{1j} - E_{2j} = E_j$, $(A_{1j} + B_{1j}F_{1,jj}C_{1j}) - (A_{2j} + B_{2j}F_{2,jj}C_{2j}) = A_j + B_jF_{jj}C_j$, $l$ is a nonnegative integer, $x_j^{(l+1)}(t_0)(j = 1, 2, \cdots, k)$ are initial conditions, and the functions $x_j^{(0)}(\cdot)$ are initial guesses. Numerical algorithms with WR suit well for parallel processing. In addition, model order reduction is another effective technique which seeks to replace a very large-scale integration system by a system of substantially lower order. There are two main kinds of model order reduction methods. The first one is the Krylov subspace method, for details see [7, 12]. The other one is the balanced truncation reduction method, e.g., see [7, 13, 16]. For nonlinear systems, some model order reduction methods are discussed in [1, 2, 15]. Some work on model order reduction can also be referred to [11, 14].

Instead of direct numerical simulation of the system (1.1), we use the Krylov subspace model order reduction technique to construct a reduced system as follows

$$\tilde{E}_j\frac{d\tilde{x}_j(t)}{dt} = \tilde{A}_j\tilde{x}_j(t) + \tilde{B}_j\tilde{u}_j(t),\ \tilde{y}_j(t) = \tilde{C}_j\tilde{x}_j(t),\ j = 1, 2, \cdots, k, \tag{1.4}$$

where $\tilde{x}_j(t) \in \mathbb{R}^{q_j}$, $\tilde{u}_j(t) \in \mathbb{R}^{m_j}$, $\tilde{y}_j(t) \in \mathbb{R}^{p_j}$, $\tilde{E}_j, \tilde{A}_j \in \mathbb{R}^{q_j\times q_j}$, $\tilde{B}_j \in \mathbb{R}^{q_j\times m_j}$, $\tilde{C}_j \in \mathbb{R}^{p_j\times q_j}$, $q_j \ll n_j$, and

$$\tilde{u}_j(t) = \sum_{i=1,i\neq j}^{k} F_{ji}\tilde{y}_i(t) + G_ju(t), \quad \tilde{y}(t) = \sum_{i=1}^{k} H_i\tilde{y}_i(t).$$

Our method combining WR with Krylov subspace seeks to remedy the shortcomings of WR, such as the poor convergence property and expensive computational costs. The WR technique not only gives a decoupling method for the system (1.1) but also offers a new model order reduction strategy based on Krylov subspace. Some concrete Krylov subspaces which can bring the original system (1.1) to the reduced system (1.4) may be constructed based on the iterative process (1.3).

The outline of this paper is organized as follows. In Section 2, we present some basic properties of solving the system (1.1), and discuss the decoupling of this system by the WR technique. Moreover, for the system of index one, the convergence condition of the WR solutions is derived. In Section 3, we reduce each independent subsystem to a system with lower order and analyze the convergence of the WR solutions for the reduced system of index one. The moment matching property is also analyzed in Section 3. In Section 4, we present a structure-preserving algorithm which preserves the differential-algebraic structure of the original system.

In Section 5, we provide some numerical experiments to show the usefulness of the new method. Finally, some conclusions are given in Section 6.

## 2. Decoupling by Waveform Relaxation

In this section, we first show some elementary results on solving time-dependent systems, and then consider decoupling of the system (1.1) by WR technique.

### 2.1. Solving time-dependent systems

We now look for the solution of the system (1.1). Let $n = n_1 + \cdots + n_k$, $m_0 = m_1 + \cdots + m_k$, $p_0 = p_1 + \cdots + p_k$, $E = \mathtt{diag}(E_1, \cdots, E_k) \in \mathbb{R}^{n\times n}$, $A = \mathtt{diag}(A_1, \cdots, A_k) \in \mathbb{R}^{n\times n}$, $B = \mathtt{diag}(B_1, \cdots, B_k) \in \mathbb{R}^{n\times m_0}$, $C = \mathtt{diag}(C_1, \cdots, C_k) \in \mathbb{R}^{p_0\times n}$, $H = \begin{bmatrix} H_1 & \cdots & H_k \end{bmatrix} \in \mathbb{R}^{p\times p_0}$, and

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^n,\ F = \begin{bmatrix} F_{11} & \cdots & F_{1k} \\ \vdots & \dots & \vdots \\ F_{k1} & \cdots & F_{kk} \end{bmatrix} \in \mathbb{R}^{m_0\times p_0},\ G = \begin{bmatrix} G_1 \\ \vdots \\ G_k \end{bmatrix} \in \mathbb{R}^{m_0\times m}.$$

Then, we can rewrite (1.1) as a block form

$$\mathcal{E}\frac{dx(t)}{dt} = \mathcal{A}x(t) + \mathcal{B}u(t),\ y(t) = \mathcal{C}x(t), \tag{2.1}$$

where $\mathcal{E} = E \in \mathbb{R}^{n\times n}$, $\mathcal{A} = A + BFC \in \mathbb{R}^{n\times n}$, $\mathcal{B} = BG \in \mathbb{R}^{n\times m}$, and $\mathcal{C} = HC \in \mathbb{R}^{p\times n}$. It should be mentioned that, one can conveniently compute the concrete form of $\mathcal{A}$ by block matrix multiplication as follows

$$\mathcal{A}=\begin{bmatrix} A_1 + B_1F_{11}C_1 & B_1F_{12}C_2 & B_1F_{13}C_3 & \cdots & B_1F_{1k}C_k \\ B_2F_{21}C_1 & A_2 + B_2F_{22}C_2 & B_2F_{23}C_3 & \cdots & B_2F_{2k}C_k \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ B_{k-1}F_{k-1,1}C_1 & \cdots & B_{k-1}F_{k-1,k-2}C_{k-2} & \ddots & B_{k-1}F_{k-1,k}C_k \\ B_kF_{k1}C_1 & \cdots & B_kF_{k,k-2}C_{k-2} & B_kF_{k,k-1}C_{k-1} & A_k + B_kF_{kk}C_k \end{bmatrix}.$$

In order to obtain the solutions of the first equation in (2.1), we need to define a matrix pencil $\{\mathcal{E}, \mathcal{A}\}$ as the one-parameter family $\{\lambda\mathcal{E} - \mathcal{A} : \lambda \in \mathbb{C}\}$ [3]. The spectrum $\sigma(\mathcal{E}, \mathcal{A})$ of the pencil is the set $\{\lambda \in \mathbb{C} : \mathtt{det}(\lambda\mathcal{E} - \mathcal{A}) = 0\}$. If $\mathtt{det}(\lambda\mathcal{E} - \mathcal{A}) \neq 0$ for some $\lambda \in \mathbb{C}$, then the matrix pencil $\{\mathcal{E}, \mathcal{A}\}$ is called regular. For the general case, if $\lambda\mathcal{E} - \mathcal{A}$ is singular for all values $\lambda$, then the system (2.1) has either no solution or infinitely many solutions for a given initial condition. Let the matrix pencil $\{\mathcal{E}, \mathcal{A}\}$ in (2.1) be regular. Then, there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$P\mathcal{E}Q = \begin{bmatrix} I_1 & 0 \\ 0 & N \end{bmatrix}, \quad P\mathcal{A}Q = \begin{bmatrix} M & 0 \\ 0 & I_2 \end{bmatrix}, \tag{2.2}$$

where $M \in \mathbb{R}^{\tau_1\times\tau_1}$ is in Jordan canonical form, $N \in \mathbb{R}^{\tau_2\times\tau_2}$ is nilpotent, and $\tau_1 + \tau_2 = n$. The matrix $M$ contains the finite eigenvalues of the matrix pencil $\{\mathcal{E}, \mathcal{A}\}$ on its diagonal and the zeros on the diagonal of the matrix $N$ represent the eigenvalues at infinity of $\{\mathcal{E}, \mathcal{A}\}$. The decomposition (2.2) is the known Weierstrass canonical form.

In the following, we discuss the existence of the solutions of the first equation in (2.1). First we define the index of the system (2.1) as the index of nilpotency of $N$, denoted by $\nu$. Then we premultiply the first equation in (2.1) by $P$ and make the coordinate transformations $\hat{x}(t) = Q^{-1}x(t),\ \hat{u}(t) = P\mathcal{B}u(t)$. After that, we partition $\hat{x}(t)$, $\hat{u}(t)$ as

$$\hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \text{ and } \hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}$$

respectively, where $\hat{u}_1(t),\ \hat{x}_1(t) \in \mathbb{R}^{\tau_1}$ and $\hat{u}_2(t),\ \hat{x}_2(t) \in \mathbb{R}^{\tau_2}$. Finally, by substituting these expressions into the first equation in (2.1), we get

$$\frac{d\hat{x}_1(t)}{dt} = M\hat{x}_1(t) + \hat{u}_1(t),\ N\frac{d\hat{x}_2(t)}{dt} = \hat{x}_2(t) + \hat{u}_2(t), \tag{2.3}$$

with the initial condition $\left[\hat{x}_1^T(t_0)\ \ \hat{x}_2^T(t_0)\right]^T = Q^{-1}x(t_0)$. It is clear that, if the initial condition $\hat{x}_2(t_0)$ satisfies the relation

$$\hat{x}_2(t_0) = -N^{\nu-1}\frac{d^{\nu-1}\hat{u}_2(t_0)}{dt^{\nu-1}} - N^{\nu-2}\frac{d^{\nu-2}\hat{u}_2(t_0)}{dt^{\nu-2}} - \cdots - N\frac{d\hat{u}_2(t_0)}{dt} - \hat{u}_2(t_0),$$

the system (2.3) has a unique continuously differentiable solution.

## 2.2. Decoupled systems and convergence analysis

We now study the WR solutions of the system (1.1) and consider the convergence of the WR solutions for the system of index one. To decouple the system (1.1), an iterative process for the $j$-th subsystem ($j = 1, 2, \cdots, k$) by WR can be constructed as follows

$$E_j\frac{dx_j^{(l+1)}(t)}{dt} = A_jx_j^{(l+1)}(t) + B_j\Big(\sum_{i=1, i\neq j}^{k} F_{ji}y_i^{(l)}(t) + F_{jj}y_j^{(l+1)}(t) + G_ju(t)\Big), \tag{2.4}$$

with the initial conditions $x_j^{(l+1)}(t_0) = x_j(t_0)$ satisfying certain consistent conditions such that the solutions of these systems exist. The functions $x_j^{(0)}(\cdot)$ are initial guesses.

To go on our analysis, we rewrite (1.2) and (2.4) as

$$\begin{cases} E_j\dfrac{dx_j^{(l+1)}(t)}{dt} = (A_j + B_jF_{jj}C_j)x_j^{(l+1)}(t) + B_j\Big(\displaystyle\sum_{i=1, i\neq j}^{k} F_{ji}C_ix_i^{(l)}(t) + G_ju(t)\Big), \\ y_j^{(l+1)}(t) = C_jx_j^{(l+1)}(t),\ y^{(l+1)}(t) = \displaystyle\sum_{i=1}^{k} H_iy_i^{(l+1)}(t). \end{cases} \tag{2.5}$$

The above iterative form for each $j$ is now independent. For the purpose at hand, we can also rewrite all the $k$ subsystems above as a block form

$$\mathcal{E}\frac{dx^{(l+1)}(t)}{dt} = \mathcal{A}_1x^{(l+1)}(t) + B(\mathcal{A}_2x^{(l)}(t) + Gu(t)),\ y^{(l+1)}(t) = \mathcal{C}x^{(l+1)}(t), \tag{2.6}$$

where $x^{(l)}(t) = \begin{bmatrix} (x_1^{(l)})^T(t) & \cdots & (x_k^{(l)})^T(t) \end{bmatrix}^T$, $\mathcal{A}_1 = \mathtt{diag}(A_1 + B_1F_{11}C_1, \cdots, A_k + B_kF_{kk}C_k)$, the matrices $\mathcal{E}$, $B$, $G$ and $\mathcal{C}$ are defined in Section 2.1, and $\mathcal{A}_2$ is defined as

$$\mathcal{A}_2 = \begin{bmatrix} 0 & F_{12}C_2 & F_{13}C_3 & \cdots & F_{1k}C_k \\ F_{21}C_1 & 0 & F_{23}C_3 & \cdots & F_{2k}C_k \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ F_{k-1,1}C_1 & \cdots & F_{k-1,k-2}C_{k-2} & 0 & F_{k-1,k}C_k \\ F_{k1}C_1 & \cdots & F_{k,k-2}C_{k-2} & F_{k,k-1}C_{k-1} & 0 \end{bmatrix}.$$

Now we analyze the convergence of the WR iteration (2.6). For simplicity, we suppose that the system (1.1) is of index one and restrict our considerations to the case when $E_j = \begin{bmatrix} I_j & 0 \\ 0 & 0 \end{bmatrix}$ in (1.1), where $I_j \in \mathbb{R}^{n_{j1}\times n_{j1}}$ is the identity matrix. For $j = 1, 2, \cdots, k$, let $n_{j2} = n_j - n_{j1}$ and $x_j^{(l)}(t) = \begin{bmatrix} (x_{j1}^{(l)})^T(t) & (x_{j2}^{(l)})^T(t) \end{bmatrix}^T$ where $x_{j1}^{(l)}(t) \in \mathbb{R}^{n_{j1}}$ and $x_{j2}^{(l)}(t) \in \mathbb{R}^{n_{j2}}$. In order to obtain the convergence conditions of the WR sequence $\{x^{(l)}(t)\}(l = 0, 1, \cdots)$, we now introduce some notations by $z_1^{(l)}(t) = \begin{bmatrix}(x_{11}^{(l)})^T(t) & \cdots & (x_{k1}^{(l)})^T(t)\end{bmatrix}^T$, $z_2^{(l)}(t) = \begin{bmatrix}(x_{12}^{(l)})^T(t) & \cdots & (x_{k2}^{(l)})^T(t)\end{bmatrix}^T$, and

$$J = \begin{bmatrix} I_{n_{11}} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I_{n_{21}} & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_{n_{k1}} & 0 \\ 0 & I_{n_{12}} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{n_{22}} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & I_{n_{k2}} \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Let $z^{(l)}(t) = \begin{bmatrix} z_1^{(l)}(t) \\ z_2^{(l)}(t) \end{bmatrix}$. It is easy to prove that $z^{(l)}(t) = Jx^{(l)}(t)$ and $J^{-1} = J^T$. Thus, the first equation in (2.6) can be transformed into

$$D\frac{dz^{(l+1)}(t)}{dt} = J\mathcal{A}_1J^{-1}z^{(l+1)}(t) + JB\mathcal{A}_2J^{-1}z^{(l)}(t) + JBGu(t), \tag{2.7}$$

where $D = \mathtt{diag}(I_1, \cdots, I_k, 0, \cdots, 0)$. For rewriting the equation above as a semi-explicit form, we partition $J\mathcal{A}_1J^{-1}, JB\mathcal{A}_2J^{-1}$ and $JBGu(t)$ as follows

$$J\mathcal{A}_1J^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad JB\mathcal{A}_2J^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad JBGu(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

where $P_{11}, Q_{11} \in \mathbb{R}^{\kappa_1\times\kappa_1}, P_{12}, Q_{12} \in \mathbb{R}^{\kappa_1\times\kappa_2}, P_{21}, Q_{21} \in \mathbb{R}^{\kappa_2\times\kappa_1}, P_{22}, Q_{22} \in \mathbb{R}^{\kappa_2\times\kappa_2}, f_1(t) \in \mathbb{R}^{\kappa_1}, f_2(t) \in \mathbb{R}^{\kappa_2}$. Here $\kappa_1 = n_{11} + \cdots + n_{k1}, \kappa_2 = n_{12} + \cdots + n_{k2}$. Therefore, the WR iteration (2.7) can be rewritten as follows

$$\begin{cases} \dfrac{dz_1^{(l+1)}(t)}{dt} = P_{11}z_1^{(l+1)}(t) + P_{12}z_2^{(l+1)}(t) + Q_{11}z_1^{(l)}(t) + Q_{12}z_2^{(l)}(t) + f_1(t), \\ 0 = P_{21}z_1^{(l+1)}(t) + P_{22}z_2^{(l+1)}(t) + Q_{21}z_1^{(l)}(t) + Q_{22}z_2^{(l)}(t) + f_2(t). \end{cases} \tag{2.8}$$

Suppose that the matrix pencils $\{E_j, A_j + B_jF_{jj}C_j\}(j = 1, 2, \cdots, k)$ are of index one, which implies that the matrix pencil $\{\mathcal{E}, \mathcal{A}_1\}$ is of index one, then the matrix $P_{22}$ is nonsingular. Thus, the algebraic equation appearing in (2.8) yields

$$z_2^{(l+1)}(t) = -P_{22}^{-1}P_{21}z_1^{(l+1)}(t) - P_{22}^{-1}Q_{21}z_1^{(l)}(t) - P_{22}^{-1}Q_{22}z_2^{(l)}(t) - P_{22}^{-1}f_2(t). \tag{2.9}$$

Substituting (2.9) into the differential part in (2.8) yields

$$\begin{aligned}\frac{dz_1^{(l+1)}(t)}{dt} =&(P_{11}-P_{12}P_{22}^{-1}P_{21})z_1^{(l+1)}(t)+(Q_{11}-P_{12}P_{22}^{-1}Q_{21})z_1^{(l)}(t)\\ &+(Q_{12}-P_{12}P_{22}^{-1}Q_{22})z_2^{(l)}(t)+f_1(t)-P_{12}P_{22}^{-1}f_2(t).\end{aligned}$$

From the equation above, we have

$$\begin{aligned}z_1^{(l+1)}(t) =&e^{R(t-t_0)}z_1^{(l+1)}(t_0)+\int_{t_0}^t e^{R(t-s)}(Q_{11}-P_{12}P_{22}^{-1}Q_{21})z_1^{(l)}(s)ds \\ &+\int_{t_0}^t e^{R(t-s)}(Q_{12}-P_{12}P_{22}^{-1}Q_{22})z_2^{(l)}(s)ds+\int_{t_0}^t e^{R(t-s)}(f_1(s)-P_{12}P_{22}^{-1}f_2(s))ds,\end{aligned} \tag{2.10}$$

where $R=P_{11}-P_{12}P_{22}^{-1}P_{21}$.

For the purpose at hand, we define operators $\mathcal{R}_1$ and $\mathcal{R}_2$ for $u\in L^2([t_0,t_0+T],\mathbb{R}^{\kappa_1})$ and $v\in L^2([t_0,t_0+T],\mathbb{R}^{\kappa_2})$ as

$$\begin{aligned}(\mathcal{R}_1u)(t)&=\int_{t_0}^t e^{R(t-s)}(Q_{11}-P_{12}P_{22}^{-1}Q_{21})u(s)ds,\\ (\mathcal{R}_2v)(t)&=\int_{t_0}^t e^{R(t-s)}(Q_{12}-P_{12}P_{22}^{-1}Q_{22})v(s)ds.\end{aligned}$$

For simplicity, we denote

$$\begin{aligned}\varphi_1(t)&=e^{R(t-t_0)}z_1^{(l+1)}(t_0)+\int_{t_0}^t e^{R(t-s)}(f_1(s)-P_{12}P_{22}^{-1}f_2(s))ds,\\ \varphi_2(t)&=-P_{22}^{-1}P_{21}(e^{R(t-t_0)}z_1^{(l+1)}(t_0)+\int_{t_0}^t e^{R(t-s)}(f_1(s)-P_{12}P_{22}^{-1}f_2(s))ds)-P_{22}^{-1}f_2(t).\end{aligned}$$

For any fixed $l$, by substituting (2.10) into (2.9), we may write (2.8) compactly as

$$\begin{bmatrix} z_1^{(l+1)}(t)\\ z_2^{(l+1)}(t)\end{bmatrix}=\mathcal{R}\begin{bmatrix} z_1^{(l)}(t)\\ z_2^{(l)}(t)\end{bmatrix}+\begin{bmatrix}\varphi_1(t)\\ \varphi_2(t)\end{bmatrix},$$

where

$$\mathcal{R}=\begin{bmatrix}\mathcal{R}_1 & \mathcal{R}_2\\ -P_{22}^{-1}P_{21}\mathcal{R}_1 & -P_{22}^{-1}P_{21}\mathcal{R}_2\end{bmatrix}+\begin{bmatrix}0 & 0\\ -P_{22}^{-1}Q_{21} & -P_{22}^{-1}Q_{22}\end{bmatrix}.$$

For convenience sake, we denote $\rho(R)$ as the spectral radius of the matrix $R$ and $\sigma(\mathcal{R})$ as the spectrum of the operator $\mathcal{R}$ respectively. For the sequence $\{z^{(l)}(t)\}(l=0,1,\cdots)$, we have a conclusion on its convergence.

**Theorem 2.1.** *For the system* (1.1), *let the matrix pencils* $\{E_j,A_j\}$ *and* $\{E_j,A_j+B_jF_{jj}C_j\}(j=1,2,\cdots,k)$ *be of index one. Then, for the WR iteration* (2.7) *the sequence* $\{z^{(l)}(t)\}(l=0,1,\cdots)$ *on* $[t_0,t_0+T]$ *is convergent if* $\rho(P_{22}^{-1}Q_{22})<1$.

*Proof.* Following the idea in [6], we show the theorem above. Let $\lambda\neq 0$. For any fixed $\begin{bmatrix} v_1^T & v_2^T\end{bmatrix}^T\in L^2([t_0,t_0+T],\mathbb{R}^{\kappa_1+\kappa_2})\bigcap C^1([t_0,t_0+T],\mathbb{R}^{\kappa_1+\kappa_2})$, suppose that

$$\left(\lambda\mathcal{I}-\begin{bmatrix}\mathcal{R}_1 & \mathcal{R}_2\\ -P_{22}^{-1}P_{21}\mathcal{R}_1 & -P_{22}^{-1}P_{21}\mathcal{R}_2\end{bmatrix}\right)\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}=\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix},$$

where $\mathcal{I}$ is the identity operator. It yields

$$\begin{cases} \lambda\xi_1(t) - \mathcal{R}_1\xi_1(t) - \mathcal{R}_2\xi_2(t) = v_1(t), \\ \lambda\xi_2(t) + P_{22}^{-1}P_{21}\mathcal{R}_1\xi_1(t) + P_{22}^{-1}P_{21}\mathcal{R}_2\xi_2(t) = v_2(t). \end{cases} \tag{2.11}$$

Let $F_1 = Q_{11} - P_{12}P_{22}^{-1}Q_{21}$ and $F_2 = Q_{12} - P_{12}P_{22}^{-1}Q_{22}$. From (2.11), we have

$$\begin{cases} \lambda\dfrac{d\xi_1(t)}{dt} - F_1\xi_1(t) - F_2\xi_2(t) - R(\mathcal{R}_1\xi_1(t) + \mathcal{R}_2\xi_2(t)) = \dfrac{dv_1(t)}{dt}, \\ \lambda\dfrac{d\xi_2(t)}{dt} + P_{22}^{-1}P_{21}F_1\xi_1(t) + P_{22}^{-1}P_{21}F_2\xi_2(t) + P_{22}^{-1}P_{21}R(\mathcal{R}_1\xi_1(t) + \mathcal{R}_2\xi_2(t)) = \dfrac{dv_2(t)}{dt}. \end{cases} \tag{2.12}$$

It is clear that the first equation in (2.11) yields $\mathcal{R}_1\xi_1(t)+\mathcal{R}_2\xi_2(t) = \lambda\xi_1(t)-v_1(t)$. Substituting this expression into (2.12), we have

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \frac{d\xi_1(t)}{dt} \\ \frac{d\xi_2(t)}{dt} \end{bmatrix} + S \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{dv_1(t)}{dt} - Rv_1(t) \\ \frac{dv_2(t)}{dt} + P_{22}^{-1}P_{21}Rv_1(t) \end{bmatrix}, \tag{2.13}$$

where

$$S = \begin{bmatrix} -(\lambda R + F_1) & -F_2 \\ P_{22}^{-1}P_{21}(\lambda R + F_1) & P_{22}^{-1}P_{21}F_2 \end{bmatrix}.$$

The solution of (2.13) can be written as

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \frac{1}{\lambda}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \frac{1}{\lambda}\int_{t_0}^{t} e^{\frac{s-t}{\lambda}S} \begin{bmatrix} -Rv_1(s) \\ P_{22}^{-1}P_{21}Rv_1(s) \end{bmatrix} ds - \frac{S}{\lambda^2}\int_{t_0}^{t} e^{\frac{s-t}{\lambda}S} \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} ds.$$

It implies that the operator

$$\lambda\mathcal{I} - \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ -P_{22}^{-1}P_{21}\mathcal{R}_1 & -P_{22}^{-1}P_{21}\mathcal{R}_2 \end{bmatrix}$$

has a bounded inverse in $L^2([t_0, t_0 + T], \mathbb{R}^{\kappa_1+\kappa_2})$ for any fixed $\lambda \neq 0$, i.e.,

$$\sigma\left(\begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ -P_{22}^{-1}P_{21}\mathcal{R}_1 & -P_{22}^{-1}P_{21}\mathcal{R}_2 \end{bmatrix}\right) = \{0\}.$$

Furthermore, for the operator $\mathcal{R}$ we have

$$\sigma(\mathcal{R}) = \sigma\left(\begin{bmatrix} 0 & 0 \\ -P_{22}^{-1}Q_{21} & -P_{22}^{-1}Q_{22} \end{bmatrix}\right).$$

This completes the proof. □

For WR solutions of dynamic systems, some discussions on convergence conditions can be found in [6] and the references therein.

## 3. Krylov Subspace Acceleration

In this section, we first present the Krylov subspace model order reduction method for time-dependent systems. Then the convergence conditions on the accelerated WR iteration are derived for the system of index one. Finally, we analyze the moment matching property.

### 3.1. Krylov subspace model order reduction

We consider the use of the Krylov subspace technique for the reduction of the system (1.1). A Krylov subspace is a linear space spanned by a sequence of vectors generated by a given matrix and a vector. Given a matrix $A$ and a starting vector $b$, the $r$-th Krylov subspace $\mathcal{K}_r(A,b)$ is spanned by a sequence of $r$ column vectors, denoted by $\mathcal{K}_r(A,b) = \mathtt{span}\{b, Ab, \cdots, A^{r-1}b\}$. If $B$ is also a matrix, we define the $r$-th block Krylov subspace $\mathcal{K}_r(A,B)$ as $\mathcal{K}_r(A,B) = \mathtt{colspan}\{B, AB, \cdots, A^{r-1}B\}$. By the Arnoldi procedure based on Krylov subspace, a column-orthonormal matrix can be constructed.

Now we present a model order reduction method based on Krylov subspace to reduce the decoupled system (2.5). For fixed $\lambda \in \mathbb{C}$ and each $j \in \{1, 2, \cdots, k\}$, let the matrix $A_j + B_jF_{jj}C_j - \lambda E_j$ be nonsingular. We first construct a column-orthonomal matrix $\mathcal{W}_j \in \mathbb{R}^{n_j \times q_j}$ based on the Krylov subspace

$$\mathcal{K}_{r_j}\Big((A_j + B_jF_{jj}C_j - \lambda E_j)^{-1}E_j,\ (A_j + B_jF_{jj}C_j - \lambda E_j)^{-1}B_j\Big), \tag{3.1}$$

where $r_j$ is a nonnegative integer and $q_j$ is the dimension of this Krylov subspace. Then, by taking the following approximations

$$x_j^{(l)}(t) \approx \mathcal{W}_j\tilde{x}_j^{(l)}(t),\ j = 1, 2, \cdots, k, \tag{3.2}$$

and substituting (3.2) into (2.5), we get

$$\begin{cases} \tilde{E}_j\dfrac{d\tilde{x}_j^{(l+1)}(t)}{dt} = (\tilde{A}_j + \tilde{B}_jF_{jj}\tilde{C}_j)\tilde{x}_j^{(l+1)}(t) + \tilde{B}_j\Big(\displaystyle\sum_{i=1,i\neq j}^{k} F_{ji}\tilde{C}_i\tilde{x}_i^{(l)}(t) + G_ju(t)\Big), \\ \tilde{y}_j^{(l+1)}(t) = \tilde{C}_j\tilde{x}_j^{(l+1)}(t),\ \tilde{y}^{(l+1)}(t) = \displaystyle\sum_{i=1}^{k} H_i\tilde{y}_i^{(l+1)}(t), \end{cases} \tag{3.3}$$

where $\tilde{E}_j = \mathcal{W}_j^TE_j\mathcal{W}_j, \tilde{A}_j = \mathcal{W}_j^TA_j\mathcal{W}_j, \tilde{B}_j = \mathcal{W}_j^TB_j, \tilde{C}_j = C_j\mathcal{W}_j$, and the initial conditions $\tilde{x}_j^{(l+1)}(t_0) = \mathcal{W}_j^Tx_j^{(l+1)}(t_0)$. The functions $\tilde{x}_j^{(0)}(\cdot)$ are initial guesses.

Let

$$u_j^{(l)}(t) = \sum_{i=1,i\neq j}^{k} F_{ji}C_ix_i^{(l)}(t) + G_ju(t),\quad \tilde{u}_j^{(l)}(t) = \sum_{i=1,i\neq j}^{k} F_{ji}\tilde{C}_i\tilde{x}_i^{(l)}(t) + G_ju(t).$$

If we regard the variables $u_j^{(l)}(t)$ and $\tilde{u}_j^{(l)}(t)$ as internal inputs of the $j$-th subsystems in (2.5) and in (3.3) respectively, the corresponding transfer functions can be defined by

$$H_j(s) = C_j(sE_j - (A_j + B_jF_{jj}C_j))^{-1}B_j,\ \tilde{H}_j(s) = \tilde{C}_j(s\tilde{E}_j - (\tilde{A}_j + \tilde{B}_jF_{jj}\tilde{C}_j))^{-1}\tilde{B}_j.$$

We expand $H_j(s)$ into a Taylor series as $H_j(s) = \sum_{i=0}^{+\infty} M_{i,j}(s-\lambda)^i$ where $M_{i,j} = -C_j((A_j + B_jF_{jj}C_j - \lambda E_j)^{-1}E_j)^i(A_j + B_jF_{jj}C_j - \lambda E_j)^{-1}B_j$. The coefficient matrices $M_{i,j}$ $(i = 0, 1, \cdots)$ are the moments of the $j$-th subsystem in (2.5) at the point $\lambda$.

For the reduced system (3.3), let the matrices $\tilde{A}_j + \tilde{B}_jF_{jj}\tilde{C}_j - \lambda\tilde{E}_j (j = 1, 2, \cdots, k)$ be nonsingular.Then, we expand $\tilde{H}_j(s)$ into a Taylor series as $\tilde{H}_j(s) = \sum_{i=0}^{+\infty} \tilde{M}_{i,j}(s-\lambda)^i$ where

$$\tilde{M}_{i,j} = -\tilde{C}_j\Big((\tilde{A}_j + \tilde{B}_jF_{jj}\tilde{C}_j - \lambda\tilde{E}_j)^{-1}\tilde{E}_j\Big)^i(\tilde{A}_j + \tilde{B}_jF_{jj}\tilde{C}_j - \lambda\tilde{E}_j)^{-1}\tilde{B}_j$$

are the moments of the $j$-th subsystem in (3.3) at the point $\lambda$. Since $\mathcal{K}_{r_j}((A_j + B_jF_{jj}C_j - \lambda E_j)^{-1}E_j, (A_j + B_jF_{jj}C_j - \lambda E_j)^{-1}B_j) \subseteq \mathtt{colspan}\{\mathcal{W}_j\}$, according to [7] we can prove $M_{i,j} = \tilde{M}_{i,j}(i = 0, 1, \cdots, r_j - 1)$. For every step of the WR iteration, we use the same Krylov subspace (3.1) to accomplish the reduction process. The matrices $\tilde{E}_j, \tilde{A}_j, \tilde{B}_j$, and $\tilde{C}_j$ are unchanged. Therefore, for every iteration of the $j$-th subsystem, $\tilde{H}_j(s)$ interpolates $H_j(s)$ at the point $\lambda$ up to the first $r_j$ moments.

It should be noted that, from the perspective of numerical simulation, the reduced system simulated by the WR technique in this paper is decomposed. The WR technique suits well for parallel computation, especially when the original problem consists of some subprocesses with many internal and few external variables, such as the system (1.1). Decoupling and parallelization of systems embedded in the WR technique are two ways to significantly reduce overall computation time. Furthermore, to reduce the block system (2.1), the Krylov subspace method requires $O(qn^2)$ flops, where $q$ is the order of the reduced system. However, since $O(q_jn_j^2)$ flops are required to construct a reduced system of order $q_j$ for the $j$-th subsystem in (2.5), the total cost of our method is $O(\sum_{j=1}^k q_jn_j^2)$. Since $n = \sum_{j=1}^k n_j$, our method is superior in numerical efficiency with cheaper calculations.

### 3.2. Convergence analysis

We now analyze convergence conditions of the sequences $\{x_j^{(l)}(t)\}(j = 1, 2, \cdots, k)$ in (3.3). First, we need to partition $\mathcal{W}_j$ as $\mathcal{W}_j = \begin{bmatrix} \mathcal{W}_{j1} \\ \mathcal{W}_{j2} \end{bmatrix}$ where $\mathcal{W}_{j1} \in \mathbb{R}^{n_{j1}\times q_j}$ and $\mathcal{W}_{j2} \in \mathbb{R}^{n_{j2}\times q_j}$. In view of $E_j = \begin{bmatrix} I_j & 0 \\ 0 & 0 \end{bmatrix}$, we can rewrite the system (3.3) as follows

$$\tilde{\mathcal{E}}\frac{d\tilde{x}^{(l+1)}(t)}{dt} = \tilde{\mathcal{A}}_1\tilde{x}^{(l+1)}(t) + \tilde{B}(\tilde{\mathcal{A}}_2\tilde{x}^{(l)}(t) + Gu(t)),\ \tilde{y}^{(l+1)}(t) = \tilde{\mathcal{C}}\tilde{x}^{(l+1)}(t), \tag{3.4}$$

where $\tilde{x}^{(l)}(t) = \begin{bmatrix} (\tilde{x}_1^{(l)})^T(t) & \cdots & (\tilde{x}_k^{(l)})^T(t) \end{bmatrix}^T$, $\tilde{\mathcal{A}}_1 = \mathtt{diag}(\tilde{A}_1+\tilde{B}_1F_{11}\tilde{C}_1, \cdots, \tilde{A}_k+\tilde{B}_kF_{kk}\tilde{C}_k)$, $\tilde{\mathcal{E}} = \mathtt{diag}(\mathcal{W}_{11}^T\mathcal{W}_{11}, \cdots, \mathcal{W}_{k1}^T\mathcal{W}_{k1})$, $\quad \tilde{B} = \mathtt{diag}(\tilde{B}_1, \cdots, \tilde{B}_k)$, $\quad \tilde{\mathcal{C}} = H\mathtt{diag}(\tilde{C}_1, \cdots, \tilde{C}_k)$, and

$$\tilde{\mathcal{A}}_2 = \begin{bmatrix} 0 & F_{12}\tilde{C}_2 & F_{13}\tilde{C}_3 & \cdots & F_{1k}\tilde{C}_k \\ F_{21}\tilde{C}_1 & 0 & F_{23}\tilde{C}_3 & \cdots & F_{2k}\tilde{C}_k \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ F_{k-1,1}\tilde{C}_1 & \cdots & F_{k-1,k-2}\tilde{C}_{k-2} & 0 & F_{k-1,k}\tilde{C}_k \\ F_{k1}\tilde{C}_1 & \cdots & F_{k,k-2}\tilde{C}_{k-2} & F_{k,k-1}\tilde{C}_{k-1} & 0 \end{bmatrix}.$$

Let the matrices $\mathcal{W}_{j1}(j = 1, 2, \cdots, k)$ be of full column rank, which implies that the matrices $\mathcal{W}_{j1}^T\mathcal{W}_{j1}$ are nonsingular. Then the matrix $\tilde{\mathcal{E}}$ is nonsingular. Therefore, we can rewrite (3.4) as an operator equation

$$\tilde{x}^{(l+1)}(t) = \mathcal{R}_3\tilde{x}^{(l)}(t) + \int_{t_0}^t e^{\tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1(t-s)}\tilde{\mathcal{E}}^{-1}\tilde{B}Gu(s)ds + e^{\tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1(t-t_0)}\tilde{x}^{(l+1)}(t_0),$$

where the operator $\mathcal{R}_3$ is defined as $(\mathcal{R}_3u)(t) = \int_{t_0}^t e^{\tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1(t-s)}\tilde{\mathcal{E}}^{-1}\tilde{B}\tilde{\mathcal{A}}_2u(s)ds$ for $u \in L^2([t_0, t_0+T], \mathbb{R}^q)$. Here $q = q_1 + q_2 + \cdots + q_k$. Thus, for the accelerated WR iteration (3.4) we have the following conclusion.

**Theorem 3.1.** *Let the matrices $\mathcal{W}_{j1}(j = 1, 2, \cdots, k)$ be of full column rank. Then, for the WR iteration (3.4), the sequence $\{\tilde{x}^{(l)}(t)\}$ on $[t_0, t_0 + T]$ is convergent.*

*Proof.* We only need to prove $\sigma(\mathcal{R}_3) = \{0\}$. Note that $\mathcal{R}_3 : L^2([t_0, t_0+T], \mathbb{R}^q) \to L^2([t_0, t_0+T], \mathbb{R}^q)$ is a bounded linear operator. Referring to Theorem 7.5-4 of [8], we know that $\sigma(\mathcal{R}_3)$ is not empty.

In the following, for any fixed $\lambda \neq 0$ we prove $\lambda \notin \sigma(\mathcal{R}_3)$. Suppose that the operator equation $\lambda u(t) - \mathcal{R}_3 u(t) = g(t)$ holds for $g \in L^2([t_0, t_0+T], \mathbb{R}^q) \bigcap C^1([t_0, t_0+T], \mathbb{R}^q)$. From this operator equation, we have

$$\lambda \frac{du(t)}{dt} - \tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1 \int_{t_0}^{t} e^{\tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1(t-s)} \tilde{\mathcal{E}}^{-1}\tilde{B}\tilde{\mathcal{A}}_2 u(s) ds - \tilde{\mathcal{E}}^{-1}\tilde{B}\tilde{\mathcal{A}}_2 u(t) = \frac{dg(t)}{dt}.$$

Substituting $\mathcal{R}_3 u(t) = \lambda u(t) - g(t)$ into the above equation, it follows

$$\frac{du(t)}{dt} - (\tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1 + \frac{1}{\lambda}\tilde{\mathcal{E}}^{-1}\tilde{B}\tilde{\mathcal{A}}_2)u(t) = \frac{1}{\lambda}(\frac{dg(t)}{dt} - \tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1 g(t)).$$

It is clear that the solution of the above equation is

$$u(t) = \frac{1}{\lambda} g(t) + \frac{1}{\lambda^2} \int_{t_0}^{t} e^{(\tilde{\mathcal{E}}^{-1}\tilde{\mathcal{A}}_1 + \frac{1}{\lambda}\tilde{\mathcal{E}}^{-1}\tilde{B}\tilde{\mathcal{A}}_2)(t-s)} \tilde{\mathcal{E}}^{-1}\tilde{B}\tilde{\mathcal{A}}_2 g(s) ds.$$

Therefore, $\lambda\mathcal{I} - \mathcal{R}_3$ has a bounded inverse in $L^2([t_0, t_0 + T], \mathbb{R}^q)$, i.e., $\lambda \notin \sigma(\mathcal{R}_3)$. Thus, for the operator $\mathcal{R}_3$ we have $\sigma(\mathcal{R}_3) = \{0\}$. This completes the proof. $\square$

Under the condition of Theorem 3.1, let the sequences $\tilde{x}^{(l)}(t)$ and $\tilde{y}^{(l)}(t)$ $(l = 0, 1, \cdots)$ in (3.4) converge to limit functions $\tilde{x}(t)$ and $\tilde{y}(t)$ respectively. Then we get

$$\tilde{\mathcal{E}} \frac{d\tilde{x}(t)}{dt} = \tilde{\mathcal{A}}\tilde{x}(t) + \tilde{\mathcal{B}}u(t), \; \tilde{y}(t) = \tilde{\mathcal{C}}\tilde{x}(t), \tag{3.5}$$

where $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1 + \tilde{B}\tilde{\mathcal{A}}_2$ and $\tilde{\mathcal{B}} = \tilde{B}G$. From the point of view of numerical simulation, the WR iteration (3.3) gives a strategy to solve $\tilde{x}(t)$ in (3.5).

It should be pointed out that, if we apply the approximation $x(t) \approx \mathcal{W}\tilde{x}(t)$ where $\mathcal{W} = \texttt{diag}(\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_k)$ to the block system (2.1), the reduced system (3.5) can be also constructed. Let $r^* = \min\{r_1, r_2, \cdots, r_k\}$. Then it has

$$\mathcal{K}_{r^*}\Big((A + B\hat{F}C - \lambda E)^{-1}E, (A + B\hat{F}C - \lambda E)^{-1}B\Big) \subseteq \texttt{colspan}\{\mathcal{W}\}, \tag{3.6}$$

where $\hat{F} = \texttt{diag}(F_{11}, F_{22}, \cdots, F_{kk})$. By the Laplace transformations, the transfer functions of the systems (2.1) and (3.5) can be given by $H(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ and $\tilde{H}(s) = \tilde{\mathcal{C}}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})^{-1}\tilde{\mathcal{B}}$ respectively. We will demonstrate the relation between $H(s)$ and $\tilde{H}(s)$ later in Section 3.3.

### 3.3. Moment matching

In Sections 3.1, we have considered two major objects of our method: (I) we decoupled the system (1.1) by the WR technique; (II) every independent subsystem has been reduced by the Krylov subspace technique. In other words, an accelerated WR process based on Krylov subspace for the system (1.1) has been constructed.

In order to approximate the global mapping from $U(s)$ and $Y(s)$ where $U(s)$ and $Y(s)$ are the Laplace transformations of the external input $u(t)$ and the external output $y(t)$ in (1.1) respectively, the objective of our work is to reduce

$$H_j(s) = C_j\Big(sE_j - (A_j + B_jF_{jj}C_j)\Big)^{-1}B_j,$$

which are obtained from the WR iteration (2.5). Let the matrices $A+BFC-\lambda E$, $A+B\hat{F}C-\lambda E$, and $A-\lambda E$ be nonsingular for fixed $\lambda \in \mathbb{C}$. Obviously, to reduce the block system (2.1), a direct model order reduction method can be based on the Krylov subspace $\mathcal{K}_r((A+BFC-\lambda E)^{-1}E, (A+BFC-\lambda E)^{-1}BG)$. However, in our work the Krylov subspace can be constructed as $\mathcal{K}_r((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B)$. For these Krylov subspaces, we have the following theorem to state their inclusion relation. For convenience sake, we define the Krylov matrix $M_r(A,B) = \begin{bmatrix} B & AB & \cdots & A^{r-1}B \end{bmatrix}$ corresponding to the Krylov subspace $\mathcal{K}_r(A,B)$.

**Theorem 3.2.** *Let the matrices $A+BFC-\lambda E$, $A+B\hat{F}C-\lambda E$, and $A-\lambda E$ be nonsingular. Then*

$$\begin{aligned}&\mathcal{K}_r\Big((A+BFC-\lambda E)^{-1}E, (A+BFC-\lambda E)^{-1}BG\Big)\\ \subseteq&\mathcal{K}_r\Big((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B\Big)\\ \subseteq&\mathcal{K}_r\Big((A-\lambda E)^{-1}E, (A-\lambda E)^{-1}B\Big).\end{aligned}$$

*Proof.* We only prove

$$\begin{aligned}&\mathcal{K}_r\Big((A+BFC-\lambda E)^{-1}E, (A+BFC-\lambda E)^{-1}BG\Big)\\ \subseteq&\mathcal{K}_r\Big((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B\Big).\end{aligned}$$

An analog proof can be given for $\mathcal{K}_r((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B) \subseteq \mathcal{K}_r((A-\lambda E)^{-1}E, (A-\lambda E)^{-1}B)$.

First, we prove $\texttt{colspan}\{(A+BFC-\lambda E)^{-1}BG\} \subseteq \texttt{colspan}\{(A+B\hat{F}C-\lambda E)^{-1}B\}$. Since

$$\begin{aligned}&(A+BFC-\lambda E)^{-1}BG\\ =&((A+B\hat{F}C-\lambda E) - B(\hat{F}-F)C)^{-1}BG\\ =&(I-(A+B\hat{F}C-\lambda E)^{-1}B(\hat{F}-F)C)^{-1}(A+B\hat{F}C-\lambda E)^{-1}BG\\ =&(A+B\hat{F}C-\lambda E)^{-1}B(I-(\hat{F}-F)C(A+B\hat{F}C-\lambda E)^{-1}B)^{-1}G,\end{aligned}$$

the column space of $(A+BFC-\lambda E)^{-1}BG$ is included in the column space of $(A+B\hat{F}C-\lambda E)^{-1}B$. Assuming that $\mathcal{K}_k((A+BFC-\lambda E)^{-1}E, (A+BFC-\lambda E)^{-1}BG) \subseteq \mathcal{K}_k((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B)$ for any $k \in \{1,2,\cdots,r-1\}$, we need to prove

$$\begin{aligned}&\mathcal{K}_{k+1}\Big((A+BFC-\lambda E)^{-1}E, (A+BFC-\lambda E)^{-1}BG\Big)\\ \subseteq&\mathcal{K}_{k+1}\Big((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B\Big).\end{aligned}\tag{3.7}$$

According to this assumption, it is clear that there exists a matrix $\Psi$ such that $((A+BFC-\lambda E)^{-1}E)^{k-1}(A+BFC-\lambda E)^{-1}BG = M_k((A+B\hat{F}C-\lambda E)^{-1}E, (A+B\hat{F}C-\lambda E)^{-1}B)\Psi$. It follows that

$$
\begin{aligned}
&\Big((A+BFC-\lambda E)^{-1}E\Big)^{k}(A+BFC-\lambda E)^{-1}BG\\
=&(A+BFC-\lambda E)^{-1}EM_k\Big((A+B\hat{F}C-\lambda E)^{-1}E,(A+B\hat{F}C-\lambda E)^{-1}B\Big)\Psi\\
=&\Big((A+B\hat{F}C-\lambda E)-B(\hat{F}-F)C\Big)^{-1}EM_k\Big((A+B\hat{F}C-\lambda E)^{-1}E,(A+B\hat{F}C-\lambda E)^{-1}B\Big)\Psi\\
=&\Big(I-(A+B\hat{F}C-\lambda E)^{-1}B(\hat{F}-F)C\Big)^{-1}(A+B\hat{F}C-\lambda E)^{-1}E\\
&\times M_k((A+B\hat{F}C-\lambda E)^{-1}E,(A+B\hat{F}C-\lambda E)^{-1}B\Big)\Psi\\
=&\sum_{i=0}^{+\infty}\Big((A+B\hat{F}C-\lambda E)^{-1}B(\hat{F}-F)C\Big)^{i}\\
&\times M_{k+1}\Big((A+B\hat{F}C-\lambda E)^{-1}E,(A+B\hat{F}C-\lambda E)^{-1}B\Big)[0\quad \Psi^T]^T.
\end{aligned}
$$

For any $i\geq 1$, it is clear that

$$\texttt{colspan}\{((A+B\hat{F}C-\lambda E)^{-1}B(\hat{F}-F)C)^{i}\}\subseteq \texttt{colspan}\{(A+B\hat{F}C-\lambda E)^{-1}B\}.$$

This implies that (3.7) is satisfied. $\square$

According to Theorem 3.2, our method can guarantee the moment matching approximation. From (3.6), we know that $\tilde{H}(s)$ interpolates $H(s)$ at the point $\lambda$ up to the first $r^*$ moments at least.

## 4. Structure-Preserving Process

The structure-preserving process has been previously studied in [11]. In this section, we discuss a structure-preserving algorithm by the WR technique, which preserves the differential-algebraic structure of the system (2.5).

For the system (2.5), we first construct a column-orthonormal matrix $\hat{\mathcal{W}}_j\in\mathbb{R}^{n_j\times q_j}$ based on the Krylov subspace (3.1) for each $1\leq j\leq k$. Then, we partition $\hat{\mathcal{W}}_j$ as $\hat{\mathcal{W}}_j=\begin{bmatrix}\hat{\mathcal{W}}_{j1}\\ \hat{\mathcal{W}}_{j2}\end{bmatrix}$ where $\hat{\mathcal{W}}_{j1}\in\mathbb{R}^{n_{j1}\times q_j}$, $\hat{\mathcal{W}}_{j2}\in\mathbb{R}^{n_{j2}\times q_j}$, and formally set

$$\widetilde{\mathcal{W}}_j=\begin{bmatrix}\hat{\mathcal{W}}_{j1} & 0\\ 0 & \hat{\mathcal{W}}_{j2}\end{bmatrix}\in\mathbb{R}^{n\times 2q_j}.$$

Thus, we can replace $\mathcal{W}_j$ by $\widetilde{\mathcal{W}}_j$ in (3.2) and still obtain a reduced-order system (3.3). Because of the differential-algebraic structure of the system (2.5) and the block structure of the matrix $\widetilde{\mathcal{W}}_j$, the reduced system (3.3) has the same structure as (2.5). For this structure-preserving algorithm, the matrix $\tilde{E}_j$ in (3.3) can be replaced by

$$\hat{E}_j=\begin{bmatrix}\hat{\mathcal{W}}_{j1}^T & 0\\ 0 & \hat{\mathcal{W}}_{j2}^T\end{bmatrix}\begin{bmatrix}I_{n_j} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\mathcal{W}}_{j1} & 0\\ 0 & \hat{\mathcal{W}}_{j2}\end{bmatrix}=\begin{bmatrix}\hat{\mathcal{W}}_{j1}^T\hat{\mathcal{W}}_{j1} & 0\\ 0 & 0\end{bmatrix}.$$

For convenience, we denote $\tilde{A}_j=\widetilde{\mathcal{W}}_j^TA_j\widetilde{\mathcal{W}}_j,\tilde{B}_j=\widetilde{\mathcal{W}}_j^TB_j,\tilde{C}_j=C_j\widetilde{\mathcal{W}}_j,\Lambda_j=A_j+B_jF_{jj}C_j-\lambda E_j,\tilde{\Lambda}_j=\tilde{A}_j+\tilde{B}_jF_{jj}\tilde{C}_j-\lambda\hat{E}_j,M_{i,j}=-C_j(\Lambda_j^{-1}E_j)^i\Lambda_j^{-1}B_j,\tilde{M}_{i,j}=-\tilde{C}_j(\tilde{\Lambda}_j^{-1}\hat{E}_j)^i\tilde{\Lambda}_j^{-1}\tilde{B}_j$. Since $\texttt{colspan}\{\widetilde{\mathcal{W}}_j\}\subseteq\texttt{colspan}\{\hat{\mathcal{W}}_j\}$, we have a conclusion on moment matching approximation.

**Theorem 4.1.** *For the system* (2.5) *and its reduced system* (3.3) *obtained by the structure-preserving algorithm, if* $\mathcal{K}_{r_j}(\Lambda_j^{-1}E_j, \Lambda_j^{-1}B_j) \subseteq \texttt{colspan}\{\widetilde{\mathcal{W}}_j\}$, *then we have* $\tilde{M}_{i,j} = M_{i,j}(i = 0, 1, \cdots, r_j - 1)$.

The proof of the above theorem can be found in [7]. For the reduced system (3.4) obtained by the structure-preserving algorithm, the matrix $\tilde{\mathcal{E}}$ is replaced by

$$\hat{\mathcal{E}} = \texttt{diag}\left(\begin{bmatrix} \hat{\mathcal{W}}_{11}^T\hat{\mathcal{W}}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{\mathcal{W}}_{21}^T\hat{\mathcal{W}}_{21} & 0 \\ 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} \hat{\mathcal{W}}_{k1}^T\hat{\mathcal{W}}_{k1} & 0 \\ 0 & 0 \end{bmatrix}\right) \in \mathbb{R}^{2q\times 2q}.$$

To analyze the convergence behavior, we denote $\tilde{x}_j^{(l)}(t) = \begin{bmatrix} (\tilde{x}_{j1}^{(l)})^T(t) & (\tilde{x}_{j2}^{(l)})^T(t) \end{bmatrix}^T$, where $\tilde{x}_{j1}^{(l)}(t), \tilde{x}_{j2}^{(l)}(t) \in \mathbb{R}^{q_j}$. Moreover, we need to introduce some notations by

$$\tilde{z}_1^{(l)}(t) = \begin{bmatrix} (\tilde{x}_{11}^{(l)})^T(t) & \cdots & (\tilde{x}_{k1}^{(l)})^T(t) \end{bmatrix}^T, \quad \tilde{z}_2^{(l)}(t) = \begin{bmatrix} (\tilde{x}_{12}^{(l)})^T(t) & \cdots & (\tilde{x}_{k2}^{(l)})^T(t) \end{bmatrix}^T,$$

and

$$\tilde{J} = \begin{bmatrix} I_{q_1} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I_{q_2} & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_{q_k} & 0 \\ 0 & I_{q_1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{q_2} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & I_{q_k} \end{bmatrix} \in \mathbb{R}^{2q\times 2q}.$$

Let $\tilde{z}^{(l)}(t) = \begin{bmatrix} \tilde{z}_1^{(l)}(t) \\ \tilde{z}_2^{(l)}(t) \end{bmatrix}$. It yields $\tilde{z}^{(l)}(t) = \tilde{J}\tilde{x}^{(l)}(t)$ and $\tilde{J}^{-1} = \tilde{J}^T$. To rewrite the first equation of the reduced system (3.4) as a semi-explicit form, we partition the matrices $\tilde{J}\tilde{\mathcal{A}}_1\tilde{J}^{-1}, \tilde{J}\tilde{B}\tilde{\mathcal{A}}_2\tilde{J}^{-1}$, and $\tilde{J}\tilde{B}\tilde{G}u(t)$ as follows

$$\tilde{J}\tilde{\mathcal{A}}_1\tilde{J}^{-1} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}, \quad \tilde{J}\tilde{B}\tilde{\mathcal{A}}_2\tilde{J}^{-1} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix}, \quad \tilde{J}\tilde{B}\tilde{G}u(t) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \end{bmatrix},$$

where $\tilde{P}_{11}, \tilde{Q}_{11}, \tilde{P}_{12}, \tilde{Q}_{12}, \tilde{P}_{21}, \tilde{Q}_{21}, \tilde{P}_{22}, \tilde{Q}_{22} \in \mathbb{R}^{q\times q}, \tilde{f}_1(t), \tilde{f}_2(t) \in \mathbb{R}^q$. From (3.4), we have

$$\begin{cases} \tilde{D}\dfrac{d\tilde{z}_1^{(l+1)}(t)}{dt} = \tilde{P}_{11}\tilde{z}_1^{(l+1)}(t) + \tilde{P}_{12}\tilde{z}_2^{(l+1)}(t) + \tilde{Q}_{11}\tilde{z}_1^{(l)}(t) + \tilde{Q}_{12}\tilde{z}_2^{(l)}(t) + \tilde{f}_1(t), \\ 0 = \tilde{P}_{21}\tilde{z}_1^{(l+1)}(t) + \tilde{P}_{22}\tilde{z}_2^{(l+1)}(t) + \tilde{Q}_{21}\tilde{z}_1^{(l)}(t) + \tilde{Q}_{22}\tilde{z}_2^{(l)}(t) + \tilde{f}_2(t), \end{cases} \tag{4.1}$$

where $\tilde{D} = \texttt{diag}(\hat{\mathcal{W}}_{11}^T\hat{\mathcal{W}}_{11}, \cdots, \hat{\mathcal{W}}_{k1}^T\hat{\mathcal{W}}_{k1})$. Let the matrices $\hat{\mathcal{W}}_{j1}(j = 1, 2, \cdots, k)$ be of full column rank, which implies that the matrix $\tilde{D}$ is nonsingular. Suppose that the matrix pencil $\{\hat{\mathcal{E}}, \tilde{\mathcal{A}}_1\}$ is of index one, then the matrix $\tilde{P}_{22}$ is nonsingular. Similar to the discussion in Section 3, for the accelerated WR iteration (4.1) we have a conclusion on its convergence.

**Theorem 4.2.** *Let the matrices* $\hat{\mathcal{W}}_{j1}(j = 1, 2, \cdots, k)$ *be of full column rank and the matrix pencil* $\{\hat{\mathcal{E}}, \tilde{\mathcal{A}}_1\}$ *be of index one. Then, for the WR iteration* (4.1) *on* $[t_0, t_0 + T]$, *the sequence* $\{\tilde{z}^{(l)}(t)\}$ *is convergent if* $\rho(\tilde{P}_{22}^{-1}\tilde{Q}_{22}) < 1$.

The proof of the theorem above is completely similar to that of Theorem 2.1.

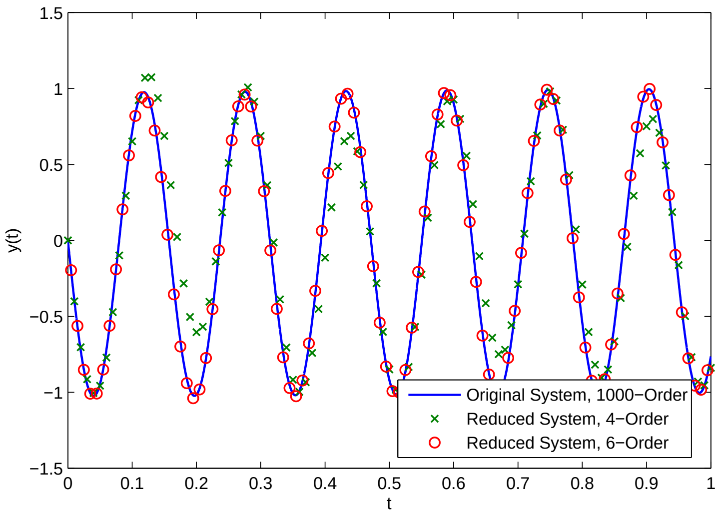


Fig. 5.1. Transient responses of Example 5.1 on WR using the Krylov subspace technique.

## 5. Numerical Experiments

In this section, we present three examples for illustration. All numerical tests are performed in Matlab environment.

**Example 5.1.** First, we consider a heated beam problem. The PI-controller is described by

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{dx_1(t)}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} k_I \\ -k_P \end{bmatrix} u_1(t), \\ y_1(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x_1(t), \end{cases} \tag{5.1}$$

where $k_I$ and $k_P$ are constants associated with the PI-controller. By a spatial discretization of the heat transfer equation along the 1D beam of length 1, it has

$$E_2 \frac{dx_2(t)}{dt} = A_2 x_2(t) + B_2 u_2(t), \;\; y_2(t) = C_2 x_2(t), \tag{5.2}$$

where $E_2 = I_{n_2}$, $B_2 = \begin{bmatrix} \kappa(n_2+1) & 0 & \cdots & 0 \end{bmatrix}^T$, $C_2 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$, and

$$A_2 = \kappa(n_2+1)^2 \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}.$$

Here, $\kappa$ is a constant and $n_2$ is a positive integer, and $u_1(t) = u(t) - y_2(t), u_2(t) = y_1(t), y(t) = y_1(t) + y_2(t)$.

It is not necessary to reduce the first subsystem because of its low order. To simulate the first subsystem, we adopt an iterative process by WR as follows

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{dx_1^{(l+1)}(t)}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_1^{(l+1)}(t) + \begin{bmatrix} k_I \\ -k_P \end{bmatrix} (u(t) - C_2 x_2^{(l)}(t)), \\ y_1^{(l+1)}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x_1^{(l+1)}(t). \end{cases} \tag{5.3}$$

Meanwhile, an iterative process on the second subsystem is given as

$$\begin{cases} E_2 \frac{dx_2^{(l+1)}(t)}{dt} = A_2 x_2^{(l+1)}(t) + B_2 \begin{bmatrix} 1 & 1 \end{bmatrix} x_1^{(l)}(t), \\ y_2^{(l+1)}(t) = C_2 x_2^{(l+1)}(t). \end{cases} \tag{5.4}$$

Before solving the equation above, we reduce its order by the Krylov subspace technique. First, a Krylov subspace is constructed by the matrix $(A_2 - 50E_2)^{-1}E_2$ and the vector $(A_2 - 50E_2)^{-1}B_2$. Then, a column-orthonomal matrix $\mathcal{W}_2 \in \mathbb{R}^{n_2 \times q_2}$ $(q_2 \ll n_2)$ can be obtained by the Arnoldi procedure. Finally, by taking an approximation $x_2^{(l+1)}(t) \approx \mathcal{W}_2 \tilde{x}_2^{(l+1)}(t)$ and substituting it into (5.4), we can get an accelerated iterative process as follows

$$\begin{cases} \tilde{E}_2 \frac{d\tilde{x}_2^{(l+1)}(t)}{dt} = \tilde{A}_2 \tilde{x}_2^{(l+1)}(t) + \tilde{B}_2 \begin{bmatrix} 1 & 1 \end{bmatrix} x_1^{(l)}(t), \\ \tilde{y}_2^{(l+1)}(t) = \tilde{C}_2 \tilde{x}_2^{(l+1)}(t), \end{cases} \tag{5.5}$$

where $\tilde{E}_2 = \mathcal{W}_2^T E_2 \mathcal{W}_2$, $\tilde{A}_2 = \mathcal{W}_2^T A_2 \mathcal{W}_2$, $\tilde{B}_2 = \mathcal{W}_2^T B_2$, $\tilde{C}_2 = C_2 \mathcal{W}_2$. For the reduced system of order 6 obtained by the Krylov subspace method, the coefficient matrices in (5.5) are calculated as $\tilde{E}_2 = I_6$, $\tilde{B}_2 = \begin{bmatrix} -118.5511 & 117.6218 & 115.3784 & 113.0146 & 123.4676 & -137.9548 \end{bmatrix}^T$, $\tilde{C}_2 = \begin{bmatrix} -2.0354\texttt{e-04} & -0.0027 & 0.0144 & -0.0366 & 0.0468 & 0.0435 \end{bmatrix}$, and

$$\tilde{A}_2 = \begin{bmatrix} -49.6311 & 98.8353 & 96.9502 & 94.9640 & 103.7474 & -115.9207 \\ 98.8353 & -246.2408 & -290.5806 & -284.6275 & -310.9534 & 347.4396 \\ 96.9502 & -290.5806 & -426.9215 & -467.1405 & -510.3474 & 570.2298 \\ 94.9640 & -284.6275 & -467.1405 & -585.9514 & -694.7603 & 776.2810 \\ 103.7474 & -310.9534 & -510.3474 & -694.7603 & -940.0964 & 1106.2624 \\ -115.9207 & 347.4396 & 570.2298 & 776.2810 & 1106.2624 & -1562.6472 \end{bmatrix}.$$

In our experiments, we use the same ODEs solver such as the Euler method to compute the WR solutions of the original system and its reduced systems. The external input is $\sin(-40t)$. Computed results are shown in Fig. 5.1. It shows that the original system is approximated very well if the order of the reduced system of the second subsystem is set to be 6. Compared with 408.32s spent on the simulation of the original system, it is about 3.16s to solve the reduced system of order 6.

For the original system and its reduced system of order 6, we now check their convergence conditions on WR. Simply, the spectral radius of the matrix $P_{22}^{-1}Q_{22}$ is zero. By Theorem 2.1, we know that the WR process for the original system is convergent theoretically. By simple calculation, we obtain $\tilde{P}_{22} = 1$ and $\tilde{Q}_{22} = 0$. Therefore, it has $\rho(\tilde{P}_{22}^{-1}\tilde{Q}_{22}) = 0$, which implies the WR processes (5.3) and (5.5) are convergent.

**Example 5.2.** Next, we consider a delay-differential system

$$\frac{dx(t)}{dt} = -x(t-1) + u(t), \quad y(t) = x(t).$$

It can be described by an interconnection of

$$\begin{cases} \frac{dx_1(t)}{dt} = \begin{bmatrix} 1 & -1 \end{bmatrix} u_1(t), & y_1(t) = x_1(t), \\ E_2 \frac{dx_2(t)}{dt} = A_2 x_2(t) + B_2 u_2(t), & y_2(t) = C_2 x_2(t), \end{cases}$$

with the coupling relation given by

$$u_1(t) = [0 \ \ 1]^T y_2(t) + [1 \ \ 0]^T u(t), \quad u_2(t) = y_1(t), \quad y(t) = y_1(t),$$

where $E_2 = I_{n_2}$, $B_2 = \begin{bmatrix} 0 & \cdots & 0 & n_2 \end{bmatrix}^T$, $C_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$, and

$$A_2 = n_2 \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}.$$

Here, $n_2$ is a large positive integer.

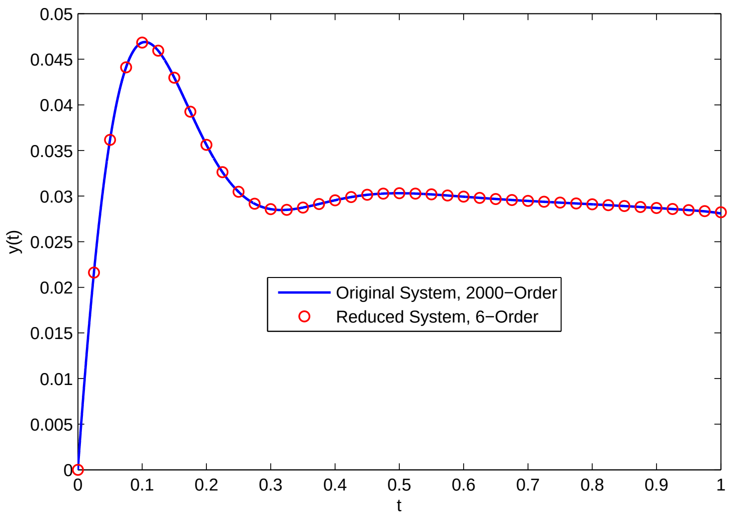


Fig. 5.2. Transient responses of Example 5.2 on WR using the Krylov subspace technique.

To simulate the first subsystem, we adopt a direct iterative process by WR as

$$\frac{dx_1^{(l+1)}(t)}{dt} = -C_2 x_2^{(l)}(t) + u(t), \quad y_1^{(l)}(t) = x_1^{(l)}(t). \tag{5.6}$$

For the second subsystem, a direct iterative process can be constructed as

$$E_2 \frac{dx_2^{(l+1)}(t)}{dt} = A_2 x_2^{(l+1)}(t) + B_2 x_1^{(l)}(t), \quad y_2^{(l)}(t) = C_2 x_2^{(l)}(t). \tag{5.7}$$

In order to reduce the order of the second subsystem, we first construct the Krylov subspace $\mathcal{K}_{r_2}(A_2^{-1}E_2, A_2^{-1}B_2)$. Then, a column-orthonomal matrix $\mathcal{W}_2 \in \mathbb{R}^{n_2 \times q_2}$ $(q_2 \ll n_2)$ is constructed by the Arnoldi procedure. Finally, by taking an approximation $x_2^{(l+1)}(t) \approx \mathcal{W}_2 \tilde{x}_2^{(l+1)}(t)$

and substituting it into (5.7), we get

$$\tilde{E}_2 \frac{d\tilde{x}_2^{(l+1)}(t)}{dt} = \tilde{A}_2 \tilde{x}_2^{(l+1)}(t) + \tilde{B}_2 x_1^{(l)}(t), \quad \tilde{y}_2^{(l)}(t) = \tilde{C}_2 \tilde{x}_2^{(l)}(t), \tag{5.8}$$

where $\tilde{E}_2 = \mathcal{W}_2^T E_2 \mathcal{W}_2, \tilde{A}_2 = \mathcal{W}_2^T A_2 \mathcal{W}_2, \tilde{B}_2 = \mathcal{W}_2^T B_2$, and $\tilde{C}_2 = C_2 \mathcal{W}_2$.

We use the Euler method to compute the WR solutions of the original system and its reduced system. The external input is $\exp(-10t)\cos(-15t) + \sinh(0.005t)$. Computed results are shown in Fig. 5.2. We also check the convergence conditions of the original WR process and the accelerated WR process. First, we can simply calculate that the spectral radius of the matrix $P_{22}^{-1} Q_{22}$ is zero. By Theorem 2.1, we know that the WR process of the original system is convergent. Then, for the reduced system of order 6 obtained by our method, the coefficient matrices in (5.8) are calculated as $\tilde{E}_2 = I_6$, and

$$\begin{aligned}
\tilde{B}_2 &= \begin{bmatrix} -33.9564 & -76.5564 & 99.8034 & -117.9870 & 133.8747 & 150.1854 \end{bmatrix}^T, \\
\tilde{C}_2 &= \begin{bmatrix} -0.0279 & 0.0384 & 0.0499 & 0.0590 & 0.0665 & -0.0721 \end{bmatrix}, \\
\tilde{A}_2 &= \begin{bmatrix}
-1.0667 & 2.1658 & 2.7844 & 3.2902 & 3.7123 & -4.0252 \\
-1.3220 & -2.9429 & -3.8387 & -4.5335 & -5.1150 & 5.5461 \\
1.6940 & 3.8254 & -4.9942 & -5.9023 & -6.6586 & 7.2198 \\
-2.0027 & -4.5198 & 5.9046 & -6.9976 & -7.8928 & 8.5570 \\
2.2723 & 5.1284 & -6.6990 & 7.9443 & -8.9944 & 9.6982 \\
2.5492 & 5.7532 & -7.5152 & 8.9110 & -10.1568 & -11.0042
\end{bmatrix}.
\end{aligned}$$

By calculation, the spectral radius of the matrix $\tilde{P}_{22}^{-1} \tilde{Q}_{22}$ is zero. According to Theorem 4.2, the WR processes (5.6) and (5.8) are convergent.

**Example 5.3.** Finally, an input-output system composed of two subsystems is considered. This system has the same form as the system (1.1). Here, $A_1 = -1$, $B_1 = C_1 = E_1 = 1$, $F_{11} = 0$, $F_{12} = G_1 = F_{21} = F_{22} = H_1 = H_2 = 1$, $G_2 = 0$, $B_2 = C_2^T = [\underbrace{0 \quad \cdots \quad 0}_{999} \quad 1]$, and

$$E_2 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -100 & 5 & & \\ 5 & -100 & \ddots & \\ & \ddots & \ddots & 5 \\ & & 5 & -100 \end{bmatrix} \in \mathbb{R}^{1000 \times 1000},$$

where $I_2 \in \mathbb{R}^{500 \times 500}$ is the identity matrix. To simulate the first subsystem, an iterative process by WR can be given as

$$\frac{dx_1^{(l+1)}(t)}{dt} = -x_1^{(l+1)}(t) + C_2 x_2^{(l)}(t) + u(t), \quad y_1^{(l+1)}(t) = x_1^{(l+1)}(t). \tag{5.9}$$

For the second subsystem, we construct the following iterative process

$$\begin{cases} E_2 \frac{dx_2^{(l+1)}(t)}{dt} = (A_2 + B_2 F_{22} C_2) x_2^{(l+1)}(t) + B_2 F_{21} C_1 x_1^{(l)}(t), \\ y_2^{(l+1)}(t) = C_2 x_2^{(l+1)}(t). \end{cases} \tag{5.10}$$

Before solving the above system, the algorithm presented in Section 4 is used to reduce the original large system to a smaller one. For this purpose, the Arnoldi procedure is used to

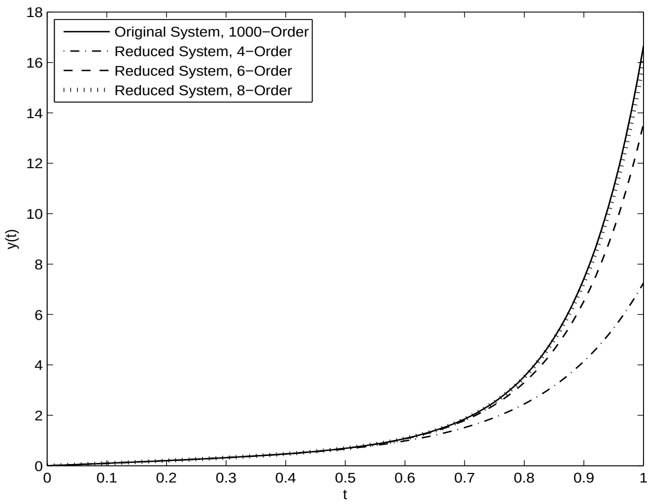


Fig. 5.3. Transient responses of Example 5.3 on WR using the Krylov subspace technique.

construct a column-orthonomal matrix $\mathcal{W}_2 \in \mathbb{R}^{1000\times q_2}$ based on the Krylov subspace $\mathcal{K}_{r_2}((A_2+B_2F_{22}C_2)^{-1}E_2, (A_2+B_2F_{22}C_2)^{-1}B_2)$. We partition $\hat{\mathcal{W}}_2$ as $\hat{\mathcal{W}}_2 = \begin{bmatrix} \hat{\mathcal{W}}_{21} \\ \hat{\mathcal{W}}_{22} \end{bmatrix}$ where $\hat{\mathcal{W}}_{21}, \hat{\mathcal{W}}_{22} \in \mathbb{R}^{500\times q_2}$. Thus, the objective of structure-preserving model order reduction can be realized by constructing the matrix $\widetilde{\mathcal{W}_2} = \mathtt{diag}(\hat{\mathcal{W}}_{21}, \hat{\mathcal{W}}_{22})$. Then the system (5.10) can be reduced to the following smaller system

$$\begin{cases} \tilde{E}_2\frac{d\tilde{x}_2^{(l+1)}(t)}{dt} = (\tilde{A}_2+\tilde{B}_2F_{22}\tilde{C}_2)\tilde{x}_2^{(l)}(t) + \tilde{B}_2F_{21}\tilde{C}_1x_1^{(l)}(t), \\ \tilde{y}_2^{(l+1)}(t) = \tilde{C}_2\tilde{x}_2^{(l+1)}(t), \end{cases} \tag{5.11}$$

where

$$\tilde{E}_2 = \widetilde{\mathcal{W}_2}^T E_2 \widetilde{\mathcal{W}_2} = \begin{bmatrix} \hat{\mathcal{W}}_{21}^T\hat{\mathcal{W}}_{21} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_2 = \widetilde{\mathcal{W}_2}^T A_2 \widetilde{\mathcal{W}_2}, \quad \tilde{B}_2 = \widetilde{\mathcal{W}_2}^T B_2, \quad \tilde{C}_2 = C_2\widetilde{\mathcal{W}_2}.$$

In our numerical simulation, we use the Euler method to compute the WR solutions of the original system and its reduced systems. The external input is the unit step function $(u(t) = 0, t \leq 0; u(t) = 1, t > 0)$. Computed results are shown in Fig. 5.3. For the original system, we can compute that the spectral radius of the matrix $P_{22}^{-1}Q_{22}$ is zero. By Theorem 2.1 it is clear that the original WR process is convergent. For the reduced system of order 8 of the second subsystem, the spectral radius of the matrix $\tilde{P}_{22}^{-1}\tilde{Q}_{22}$ is also zero. According to Theorem 4.2, the WR processes (5.9) and (5.11) are also convergent in theory.

## 6. Conclusions

In this paper, we derived an accelerated WR method based on Krylov subspace for time-dependent systems. Moreover, the convergence conditions on WR and the moment matching property have been also analyzed. As an accelerated simulation method, it is very efficient to simulate large time-dependent systems whose output behavior can be approximated very well by means of matching some moments. The method of treating such systems reported here is also useful for more complicated systems in engineering applications.

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