Journal of Computational Mathematics
Vol.31, No.3, 2013, 271–282.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1212-m3897

APPROXIMATION OF NONCONFORMING QUASI-WILSON ELEMENT FOR SINE-GORDON EQUATIONS*

Dongyang Shi
Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China
Email: shi_dy@zzu.edu.cn
Ding Zhang
School of Mathematics and Statistics, Xuchang University, Xuchang 461000, China
Email: zhangd111@126.com

Abstract

In this paper, nonconforming quasi-Wilson finite element approximation to a class of nonlinear sine-Gordan equations is discussed. Based on the known higher accuracy results of bilinear element and different techniques from the existing literature, it is proved that the inner product $(\nabla(u - I_h^1 u), \nabla v_h)$ and the consistency error can be estimated as order $O(h^2)$ in broken $H^1 - norm/L^2 - norm$ when $u \in H^3(\Omega)/H^4(\Omega)$, where $I_h^1 u$ is the bilinear interpolation of u, v_h belongs to the quasi-Wilson finite element space. At the same time, the superclose result with order $O(h^2)$ for semi-discrete scheme under generalized rectangular meshes is derived. Furthermore, a fully-discrete scheme is proposed and the corresponding error estimate of order $O(h^2 + \tau^2)$ is obtained for the rectangular partition when $u \in H^4(\Omega)$, which is as same as that of the bilinear element with ADI scheme and one order higher than that of the usual analysis on nonconforming finite elements.

Mathematics subject classification: 65N15, 65N30.
Key words: Sine-Gordon equations, Quasi-Wilson element, Semi-discrete and fully-discrete schemes, Error estimate and superclose result.

1. Introduction

Consider the following nonlinear sine-Gordon equations [1]:

$$\begin{cases} u_{tt} + \alpha u_t - \gamma \Delta u + \beta \sin u = f, & (X,t) \in \Omega \times (0,T], \\ u\mid_{\partial\Omega} = 0, & t \in (0,T], \\ u(X,0) = u_0(X), \quad u_t(X,0) = u_1(X), & X \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a convex and bounded region with Lipschitz boundary $\partial\Omega$, $X = (x,y)$, $u = u(X,t)$, α, β, γ are positive constants, $u_0, u_1, f = f(X,t)$ are known smooth functions.

There have been a lot of studies devoted to (1.1). For example, [1] proved the existence and uniqueness of the solution; [2] presented an explicit finite difference method for the numerical solution; [3] obtained analytical solutions to the unperturbed sine-Gordon equation with zero damping, i.e., $\alpha \equiv 0$; [4] established the Fourier quasi-spectrum explicit scheme and gave the convergence and error estimation; [5] discussed two implicit difference schemes and provided the numerical results; [6] studied an ADI scheme of bilinear element and deduced $O(\tau^2 + h^2)$ order estimates; [7] considered the general approximation scheme for a class of low order

* Received October 6, 2011 / Revised version received November 19, 2012 / Accepted December 25, 2012 / Published online May 6, 2013 /

nonconforming finite elements satisfying some assumptions and derived the optimal order error estimates.

As we know, Wilson element [8,9] has been widely used in engineering computation, but it is only convergent for rectangular and parallelogram meshes. In order to extend this element to arbitrary quadrilateral meshes, various improved Wilson elements were developed. For instance, [10] proposed a quasi-Wilson element by simply adding a high order term, which is independent of the element geometry, to the nonconforming part of the shape function; [11] generalized the result of [10] to a class of quasi-Wilson arbitrary quadrilateral elements; In [12], a special property is discovered, i.e., the consistency error is of order $O(h^2)$ in broken H^1-norm, one order higher than that of its interpolation error $O(h)$, which is similar to the famous nonconforming rectangular EQ_1^{rot} element[13,14], the Q_1^{rot} square element[15,16] and the constrained Q_1^{rot} element[17,18]; Especially, [19] applied this quasi-Wilson element to second-order problems on narrow quadrilateral meshes. However, all of the above studies on quasi-Wilson element are only limited to the linear problems.

In this paper, as a continuous work of [7], we will apply this quasi-Wilson finite element to problem (1.1). Based on the known higher accuracy results of bilinear element and different approaches from the existing literature, we prove that the estimations of the inner product of gradients of the difference between u and its bilinear interpolation $I_h^1 u$ with any polynomial of the finite element space and the consistency error are of order $O(h^2)$ in broken H^1-norm or L^2-norm when $u\in H^3(\Omega)$ or $H^4(\Omega)$ (see Lemmas 2.2–2.3 below). At the same time, the superclose result with order $O(h^2)$ is obtained although the mean values of quasi-Wilson element across the edges between elements are not continuous which does not satisfy the requirement (III) of [7]. Furthermore, a kind of fully-discrete scheme is proposed, and the $O(h^2+\tau^2)$ order error estimate is derived, which improves the result $O(h+\tau^2)$ of [7] by one order with respect to h, and as same as [6] with ADI scheme.

The rest of the paper is organized as follows. In the next section, we introduce the nonconforming quasi-Wilson element, and prove the important characters of the element, and derive the superclose result. In Section 3, a kind of fully-discrete scheme is proposed and the optimal order error estimate is gained.

Throughout this paper, c denotes a general positive constant which is independent of h, where $h=max_K h_K$, h_K is the diameter of the element K, τ is the time step for the partition of the time interval $[0,T]$.

2. Quasi-Wilson Element and Superclose Result

Let $\hat{K}=[0,1]\times[0,1]$ be the reference element with vertices $\hat{M}_1(0,0)$, $\hat{M}_2(1,0)$, $\hat{M}_3(1,1)$, $\hat{M}_4(0,1)$. We define on $\hat{K}$ the finite element $(\hat{K},\hat{P},\hat{\Sigma})$ as follows:

$$\hat{P}=span\Big\{N_i(\xi,\eta),(i=1,2,3,4),\hat{\Psi}(\xi),\hat{\Psi}(\eta)\Big\},$$

where $N_1(\xi,\eta)=(1-\xi)(1-\eta)$, $N_2(\xi,\eta)=\xi(1-\eta)$, $N_3(\xi,\eta)=\xi\eta$, $N_4(\xi,\eta)=(1-\xi)\eta$, $\hat{\Psi}(s)=-\frac{3}{4}s(s-1)+\frac{5}{32}[(2s-1)^4-1]$. The degrees of freedom are taken as $\hat{\Sigma}=\{\hat{v}_1,\cdots,\hat{v}_4,\hat{\beta}_1,\hat{\beta}_2\}$, where $\hat{v}_i=\hat{v}(\hat{M}_i)$, $i=1,2,3,4$. $\hat{\beta}_1=\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi^2}d\xi d\eta$, $\hat{\beta}_2=\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\eta^2}d\xi d\eta$. We have

$$\hat{v}(\xi,\eta)=\sum_{i=1}^{4}\hat{v}_iN_i(\xi,\eta)+\hat{\beta}_1\hat{\Psi}(\xi)+\hat{\beta}_2\hat{\Psi}(\eta).$$