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ON CONVERGENCE PROPERTY OF THE LANCZOS METHOD FOR SOLVING A COMPLEX SHIFTED HERMITIAN LINEAR SYSTEM*

Guiding Gu
Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China
Email: guiding@mail.shufe.edu.cn

Abstract

We discuss the convergence property of the Lanczos method for solving a complex shifted Hermitian linear system $(\alpha I + H)x = f$. By showing the colinear coefficient of two system's residuals, our convergence analysis reveals that under the condition $Re(\alpha) + \lambda_{min}(H) > 0$, the method converges faster than that for the real shifted Hermitian linear system $(Re(\alpha)I + H)x = f$. Numerical experiments verify such convergence property.

Mathematics subject classification: 65F10, 65Y20.
Key words: Hermitian matrix, Complex shifted linear system, Lanczos method.

1. Introduction

We are interested in the iterative solution of the following complex shifted Hermitian linear system

$$(\alpha I + H)x = f, \tag{1.1}$$

where $H \in C^{n\times n}$ is a Hermitian matrix, α is a complex number, called shift. Below, the linear system (1.1) will be termed the CSH linear system. Such shifted linear system arises in a variety of practical applications and has been discussed for many years; see [6-7,9,12,15]. The fundamental work of Faber and Manteuffel [8] ensured that the Arnoldi recurrence simplifies, yielding an optimal short-term recurrence. The issue was further explored and some theoretical results have been derived in [10-11,16-17].

Our motivation to discuss (1.1) comes from the HSS iteration method [5]; see also [1-2,4]. We know that in the HSS iteration method, two shifted linear sub-systems as inner iteration have to be solved per iteration step. In [14], a complex parameter α in the HSS iteration method is employed and the HSS iteration method with a suitable nonreal parameter has a smaller spectral radius of the iteration matrix than with a real parameter, even than with the experimental optimal real parameter; see also the numerical experiment 2 in this paper. Since a complex parameter α is employed in HSS, two linear sub-systems are the complex shifted linear systems, not the real shifted linear systems. In such case, an interesting discussion on the convergence rate of the shifted linear sub-system (*i.e.*, the linear system (1.1)) in the HSS iteration method is which is better, nonreal shift α, or real shift α.

In this paper, we discuss the convergence property of the Lanczos method for solving the CSH linear system (1.1). By showing the colinear coefficient of two system's residuals, our convergence analysis reveals that under the condition $Re(\alpha)+\lambda_{min}(H) > 0$, the method converges

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for the CSH linear system (1.1) faster than that for the real shifted Hermitian system

$$(Re(\alpha)I + H)x = f, \tag{1.2}$$

which indicates that the method benefits from the imaginary part of a complex shift α when solving the CSH linear system (1.1). It is known that for solving the system (1.2), the Lanczos method is equivalent to the CG method since the coefficient matrix $Re(\alpha)I + H$ is Hermitian positive definite under the condition $Re(\alpha)+\lambda_{min}(H) > 0$. This demonstrates that the Lanczos method for the CSH linear system (1.1) has a faster convergence than the CG method for the real shifted system (1.2). Numerical experiments verify such convergence property.

2. Lanczos Method and Its Convergence Property

In this section we briefly describe the direct version of the Lanczos method for solving the CSH linear system (1.1), and then give an analysis on the convergence property of the method.

The main ingredient of the Lanczos method is the following Lanczos procedure applied to the Hermitian matrix H with $v_1 = r_0/\|r_0\|_2$ as the starting vector:

For $j = 1, \cdots, m$, *Do*

$$\begin{aligned} &w_j = Hv_j - \beta_j v_{j-1}, && (if \quad j = 1,\ let \quad \beta_1 v_0 = 0)\\ &\alpha_j = (w_j, v_j), &&\\ &w_j = w_j - \alpha_j v_j, &&\\ &\beta_{j+1} = \|w_j\|, && (if \quad \beta_{j+1} = 0,\ stop.)\\ &v_{j+1} = w_j/\beta_{j+1}. && \end{aligned}$$

EndDo

We refer to [18] for a detailed description about the Lanczos procedure. By setting $V_m = [v_1, \cdots, v_m]$ and a symmetric tridiagonal matrix $T_m = tridiag(\beta_i, \alpha_i, \beta_{i+1})$, where $\beta_i, \alpha_i, \beta_{i+1}$ are the entries in the ith row of the matrix T_m, we have the shifted factorization

$$(\alpha I + H)V_m = V_m(\alpha I + T_m) + \beta_{m+1}v_{m+1}e_m^T, \tag{2.1}$$

where V_m is an orthonormal basis of $K_m \equiv K_m(H, v_1)$. Note that the Krylov subspace keeps shift invariance $K_m(H, v_1) = K_m(\alpha I + H, v_1)$. An approximation to the solution of (1.1) in $\{x_0\} + K_m$ can be written as $x_m = x_0 + V_m y_m$ and its residual is

$$\begin{aligned} r_m =& f - (\alpha I + H)x_m\\ =& r_0 - V_m(\alpha I + T_m)y_m - \beta_{m+1}v_{m+1}e_m^T y_m. \end{aligned} \tag{2.2}$$

2.1. Lanczos method

By imposing the Galerkin condition $r_m \perp K_m$, we have

$$(\alpha I + T_m)y_m = V_m^H r_0 = \beta e_1, \tag{2.3}$$

where $\beta = \|r_0\|$. Thus the Lanczos solution of the CSH linear system (1.1) has the form $x_m = x_0 + V_m(\alpha I + T_m)^{-1}(\beta e_1)$. Note that the shifted matrix $\alpha I + T_m$ is nonsingular under the condition $Re(\alpha) + \lambda_{min}(H) > 0$ in the Theorem 2.2. Let $y_m = (\eta_1, \cdots, \eta_m)^T$. By (2.2), we have the following result [18]; see also [13].