Journal of Computational Mathematics
Vol.31, No.3, 2013, 283–307.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1212-m4174

ERROR ESTIMATE ON A FULLY DISCRETE LOCAL DISCONTINUOUS GALERKIN METHOD FOR LINEAR CONVECTION-DIFFUSION PROBLEM*

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Abstract

In this paper we present the error estimate for the fully discrete local discontinuous Galerkin algorithm to solve the linear convection-diffusion equation with Dirichlet boundary condition in one dimension. The time is advanced by the third order explicit total variation diminishing Runge-Kutta method under the reasonable temporal-spatial condition as general. The optimal error estimate in both space and time is obtained by aid of the energy technique, if we set the numerical flux and the intermediate boundary condition properly.

Mathematics subject classification: 65M15, 65M60.
Key words: Runge-Kutta, Local discontinuous Galerkin method, Convection-diffusion equation, Error estimate.

1. Introduction

In this paper we shall present the a priori error estimate for one fully discrete algorithm to solve linear convection-diffusion problem with Dirichlet boundary condition. The scheme under consideration in this paper, which is referred to as the LDGRK3 scheme, uses the third order explicit total variation diminishing Runge-Kutta (TVDRK3) time-marching [19] and the local discontinuous Galerkin (LDG) spatial discretization with piecewise polynomials of arbitrary degree $k \geq 1$.

This type of method was introduced by Cockburn and Shu in [10] as an extension to general convection-diffusion problems of the numerical scheme for the compressible Navier-Stokes equation [2], which is a remarkable development from the famous Runge-Kutta discontinuous Galerkin (RKDG) methods for purely hyperbolic problems. After that, this method has been rigorously studied by a number of researchers, for example, for elliptic problem [1, 6], Stokes problem [9], and convection-diffusion problem [7]. For a fairly complete set of references on this method as well as its implementation and applications, please refer to the recent review papers [11, 18] and book [13].

To put our result in proper perspective, let us briefly describe the relevant results available in the current literature. In [10], the semi-discrete LDG method for convection diffusion problems with periodic boundary conditions was considered, and the quasi-optimal error estimate was obtained (namely k-th order accuracy in L^2-norm if the piecewise polynomials of degree k are used). Later, the convergence properties and the optimal error estimate of the hp-version of

* Received May 17, 2012 / Revised version received November 27, 2012 / Accepted December 25, 2012 / Published online May 6, 2013 /

the semi-discrete LDG method has been studied in [7], for convection-diffusion problems with Dirichlet boundary condition.

However, as far as the authors know, there are few analysis for the fully-discrete version of the LDG method. Since we actually care about the convection-dominated case in this paper, we are interesting in the explicit time-marching. We would like to adopt the explicit TVDRK3 time-marching which has a strong stability preserving (SSP) property [12] and high order accuracy in time. Based on the works of [7,22], we will devote to obtaining, for this fully-discrete method, the optimal error estimates in both time and space, for the convection-diffusion problem with Dirichlet boundary condition. In this paper we abandon the second order total variation diminishing Runge-Kutta (TVDRK2) time-marching, since it requires a more strict restriction on the temporal-spatial condition, such as $\tau = \mathcal{O}(h^{4/3})$, for the high-order piecewise polynomials, where h and τ are the maximum cell size and the time step respectively; see [3,20].

The difficulty of analysis in this paper mainly lies in two points. One is how to define nice numerical flux for the general Dirichlet boundary condition other than the periodic boundary condition. It is well known that the numerical flux is an important issue to ensure the success for LDG methods. The numerical flux is easy to implement for periodic boundary conditions, but not for Dirichlet boundary condition. In this paper we would like to follow the idea in [7], and only make some minor modifications on the numerical flux for periodic boundary condition at the boundary points. We want to find a uniform setting for numerical flux to solve out the convection-diffusion problem, in whatever case that the problem is convection-dominated or not.

The other difficulty comes from the boundary treatment at each intermediate stage time. Since the higher order TVDRK3 time marching is made up of the first-order Euler forward time-marching in each stage, incorrect boundary condition treatments may destroy the high accuracy of the LDGRK3 algorithm as that for periodic boundary condition. An important development on this reduction of convergence order has been given by Carpenter and his colleague [4], where some corrections on the intermediate boundary condition is presented for the finite difference methods to solve the hyperbolic equation. In this paper we would like to seek a reasonable treatment on the intermediate boundary condition for convection-diffusion problem from the viewpoint of energy analysis. The strategies presented here are very similar as that given in [4], which are solely based on the physical information of the given boundary condition. Several numerical experiments are also given to show the validation of our strategy on the boundary condition treatment.

The main analysis tool in this paper is the energy technique, which has been used in [20–22] to analyze the fully discrete algorithm of DG method with Runge-Kutta time-marching. This technique has many advantages in the numerical analysis. It does not demand the used mesh in uniform size, and can be extended to problems with varying-coefficients even to the nonlinear problems, it also works well for different types of boundary conditions [14]. Furthermore, it helps us to find out the reasonable and good treatment on the numerical boundary condition [15].

The remainder of this paper is organized as follows. In section 2 we present the LDGRK3 scheme for a model problem. In section 3, we give some preliminaries for the discontinuous finite element space, including the inverse properties and approximation properties for two local Gauss-Radau projections. Then we present some elemental properties of the corresponding LDG spatial discretization. Section 4 is the main body of this paper where the main result on error estimate is presented and proved by energy technique. In this process, the numerical flux and intermediate boundary conditions will be defined well. Several proofs of some basic lemmas are presented in section 7, as appendix. In section 5, some numerical experiments are given to verify the theory results and illustrate the effects of the intermediate boundary conditions. Finally, the conclusion remarks are given in section 6.

2. LDGRK3 Scheme

In this section we follow [7, 10] and present the precise definition of the LDGRK3 scheme for the model problem in one dimension

$$U_t + cU_x - dU_{xx} = 0, \qquad (x,t) \in Q_T = (a,b) \times (0,T], \tag{2.1a}$$

$$U(x,0) = U_0(x), \qquad x \in \Omega = (a,b), \tag{2.1b}$$

$$U(a,t) = U_a(t), \quad U(b,t) = U_b(t), \qquad t \in (0,T], \tag{2.1c}$$

where the constant $d > 0$ is diffusion coefficient and the constant c is the velocity of the flow field. Assume $c > 0$ in this paper, hence the location of the possible boundary layer [17] is at $x = b$. The initial solution $U_0(x)$ is assumed to be smooth enough to ensure the solution of this model problem exists uniquely with sufficient smoothness.

Let $Q = \sqrt{d}U_x$ and define $(h_U, h_Q) := (cU - \sqrt{d}Q, -\sqrt{d}U)$. The LDG scheme is started from the following equivalent first-order differential system

$$U_t + (h_U)_x = 0, \quad Q + (h_Q)_x = 0, \qquad (x,t) \in Q_T, \tag{2.2}$$

with the same initial condition (2.1b) and boundary condition (2.1c). For convenience, we denote by $\boldsymbol{W} = (U, Q)$ the exact solution of this system.

2.1. Semi-discrete LDG scheme

Let $\mathcal{T}_h = \{I_j\}_{j=1}^N$ be the quasi-uniform partition of domain Ω, where $I_j = (x_{j-1}, x_j)$ is a cell with length $h_j = x_j - x_{j-1}$ for $j = 1, \ldots, N$. Here $x_0 = a$ and $x_N = b$ are boundary endpoints. Denote $h = \max_j h_j \le 1$. Since the mesh is quasi-optimal, there exists a positive constant ν such that $h/h_j \le \nu, \forall j = 1, \ldots, N$, as h goes to zero.

Associated with the mesh $\mathcal{T}_h$, we define the discontinuous finite element space

$$V_h = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j),\ \forall j = 1, \ldots, N \right\}, \tag{2.3}$$

where $\mathcal{P}_k(I_j)$ denotes the space of polynomials in I_j of degree at most $k \ge 1$. Note that the functions in this space are allowed to have discontinuities across element interfaces. At each element interface point, for any function p, there are two traces along the right-hand and left-hand, denoted by p^+ and p^-, respectively. As usual, the jump is denoted by $[\![p]\!] = p^+ - p^-$.

We would like to seek the numerical solution, denoted by $\boldsymbol{w}(t) := (u(t), q(t))$, in the finite element space $V_h \times V_h$, where the argument x is omitted. The semi-discrete LDG scheme is defined as follows: for any $t > 0$, $\boldsymbol{w}(t)$ satisfies the variation form

$$(u_t, v)_j = \mathcal{H}_j(\boldsymbol{w}, \boldsymbol{z}) = (h_u, v_x)_j - \widehat{h}_{u,j} v_j^- + \widehat{h}_{u,j-1} v_{j-1}^+, \tag{2.4a}$$

$$(q, r)_j = \mathcal{K}_j(\boldsymbol{w}, \boldsymbol{z}) = (h_q, r_x)_j - \widehat{h}_{q,j} r_j^- + \widehat{h}_{q,j-1} r_{j-1}^+, \tag{2.4b}$$

in each cell $I_j, j = 1, 2, \ldots, N$, for any test functions $\boldsymbol{z} = (v, r) \in V_h \times V_h$, where $(\cdot,\cdot)_j$ is the usual inner product in $L^2(I_j)$. The initial condition $u(x, 0) \in V_h$ can be taken as any approximation of the given initial solution $U_0(x)$, for example, the local Gauss-Radau projection of $U_0(x)$. Please refer to (3.7) for the precise definition.

In (2.4), $\widehat{h_u}$ and $\widehat{h_q}$ are the numerical fluxes defined at every element boundary point. They depend on two values besides both sides of u and q. In this paper we would like to define them, in the similar way as that in [7]. For notational convenience, we introduce the ghost values at two boundary points

$$(u_0^-, q_0^-) = (g_a, q_0^+), \quad (u_N^+, q_N^+) = (g_b, q_N^-), \tag{2.5}$$

where $g_a = U_a$ and $g_b = U_b$ are the given Dirichlet boundary conditions. Then we define the numerical flux at x_j in the general form

$$(\widehat{h}_{u,j}, \widehat{h}_{q,j}) = (cu_j^- - \sqrt{d}q_j^+, -\sqrt{d}u_j^-) - (\gamma_j c[\![u]\!]_j, \rho_j\sqrt{d}[\![u]\!]_j), \quad j = 0, 1, \ldots, N, \tag{2.6}$$

with nonnegative constants γ_j and ρ_j.

Actually, the first term in definition (2.6) comes from the numerical flux which works well for the periodic boundary condition. The last term in (2.6) is necessary to deal with the Dirichlet boundary condition. We would like in this paper to set both γ_j and ρ_j as zero at every element boundary points, except at the boundary points of the computation domain. Note that, it is enough to set all of them to be zero for the periodic boundary condition.

The left four parameters, $\gamma_0, \gamma_N, \rho_0$ and ρ_N, need to be defined carefully to ensure the stability and accuracy of the LDG scheme, when Dirichlet boundary condition is considered. In this paper we shall seek the nice setting for these parameters from the view point of optimal error estimate. The detailed evaluations for these parameters are almost the same as that in [7]. Some of them may depend on the Péclet mesh number P_{e}, defined as

$$P_{\mathrm{e}} = \frac{ch}{d}. \tag{2.7}$$

If there holds $P_{\mathrm{e}}^{-1} = \mathcal{O}(1)$ for a given mesh, the considered problem is said to be convection-dominated.

Till now we complete the definition of the semi-discrete LDG scheme.

At the end of this subsection, we write the above scheme in the compact form for convenience. To that end, we denote by $(q, r)_h = \sum_{j=1}^{N}(q, r)_j$ the usual inner product in $L^2(\Omega)$, and by $\langle q, r\rangle_h = \sum_{j=1}^{N-1} q_j r_j$ the so-called L^2-inner product on all interior mesh grids. Summing up variation formulations (2.4) over $j = 1, 2, \ldots, N$, we can write the semi-discrete LDG scheme in the global form: for any $t > 0$, find the numerical solution $\boldsymbol{w} = (u, q) \in V_h \times V_h$ such that

$$(u_t, v)_h = \mathcal{H}(\boldsymbol{g}; \boldsymbol{w}, v) := \mathcal{H}_{\mathrm{int}}(\boldsymbol{w}, v) + \mathcal{H}_{\mathrm{bry}}(\boldsymbol{g}; v), \tag{2.8a}$$

$$(q, r)_h = \mathcal{K}(\boldsymbol{g}; u, r) := \mathcal{K}_{\mathrm{int}}(u, r) + \mathcal{K}_{\mathrm{bry}}(\boldsymbol{g}; r), \tag{2.8b}$$

hold for any test function $(v, r) \in V_h \times V_h$. Here $\boldsymbol{g} = (g_a, g_b)$ reflects the boundary condition, and the LDG spatial discretizations are given as follows:

$$\begin{aligned}\mathcal{H}_{\mathrm{int}}(\boldsymbol{w}, v) = {} & (cu - \sqrt{d}q, v_x)_h + \langle cu^- - \sqrt{d}q^+, [\![v]\!]\rangle_h - c\left[(1+\gamma_N)u_N^- v_N^- + \gamma_0 u_0^+ v_0^+\right] \\ & + \sqrt{d}(q_N^- v_N^- - q_0^+ v_0^+),\end{aligned} \tag{2.9a}$$

$$\mathcal{K}_{\mathrm{int}}(u, r) = (-\sqrt{d}u, r_x)_h - \langle\sqrt{d}u^-, [\![r]\!]\rangle_h + \sqrt{d}\left[(1-\rho_N)u_N^- r_N^- - \rho_0 u_0^+ r_0^+\right], \tag{2.9b}$$

$$\mathcal{H}_{\mathrm{bry}}(\boldsymbol{g}; v) = c\left[\gamma_N g_b v_N^- + (1+\gamma_0) g_a v_0^+\right], \tag{2.9c}$$

$$\mathcal{K}_{\mathrm{bry}}(\boldsymbol{g}; r) = \sqrt{d}\left[\rho_N g_b r_N^- - (1-\rho_0) g_a r_0^+\right]. \tag{2.9d}$$

Note the subscripts “int” and “bry” are used to emphasize the inner of domain (a, b), and the domain boundary points $x = a$ and $x = b$, respectively.

2.2. Fully discrete LDG scheme

In this paper we would like to adopt the high-order explicit TVDRK3 time-marching [19] to update the semi-discrete LDG scheme (2.4) or (2.8). This forms the fully discrete LDG scheme, called LDGRK3 scheme in this paper.

Let $\{t^n = n\tau\}_{n=0}^{M}$ be the uniform partition of time interval $[0,T]$, with the time step τ. The time step could actually change from step to step, but in this paper we take the time step as a constant for simplicity.

Then the numerical solution of LDGRK3 scheme is given by the following process. Take the initial solution $u^0 = u(x,0)$, as the same as that in the semi-discrete scheme. Assume $u^n \in V_h$ is obtained at the time t^n, we would like to find out the solution $u^{n+1} \in V_h$ at the next time t^{n+1}, through two intermediate solutions $u^{n,1} \in V_h$ and $u^{n,2} \in V_h$. For any test function $v \in V_h$, the numerical solutions satisfy that

$$(u^{n,1}, v)_h = (u^n, v)_h + \tau\mathcal{H}(\boldsymbol{g}^n; \boldsymbol{w}^n, v), \tag{2.10a}$$

$$(u^{n,2}, v)_h = \frac{3}{4}(u^n, v)_h + \frac{1}{4}(u^{n,1}, v)_h + \frac{\tau}{4}\mathcal{H}(\boldsymbol{g}^{n,1}; \boldsymbol{w}^{n,1}, v), \tag{2.10b}$$

$$(u^{n+1}, v)_h = \frac{1}{3}(u^n, v)_h + \frac{2}{3}(u^{n,2}, v)_h + \frac{2\tau}{3}\mathcal{H}(\boldsymbol{g}^{n,2}; \boldsymbol{w}^{n,2}, v), \tag{2.10c}$$

in which the auxiliary solutions $q^{n,\ell} \in V_h$ is determined by the variation form

$$(q^{n,\ell}, r)_h = \mathcal{K}(\boldsymbol{g}^{n,\ell}; u^{n,\ell}, r), \quad \forall r \in V_h, \quad \ell = 0, 1, 2. \tag{2.10d}$$

Here the notation $\boldsymbol{g}^{n,\ell} = (g_a^{n,\ell}, g_b^{n,\ell})$ is used to represent the boundary conditions at $x = a$ and $x = b$, on each intermediate time level $t^{n,\ell}$. Note that if the stage index $\ell = 0$ we would like to drop this superscript here and after; for example, $q^{n,0} = q^n$. Now we complete the definition of the considered LDGRK3 scheme.

To ensure the numerical stability of the LDGRK3 scheme, we must demand the time step τ satisfy the following temporal-spatial conditions

$$\frac{c\tau}{h} \le \lambda_{\mathrm{c}}, \quad \text{and} \quad \frac{d\tau}{h^2} \le \lambda_{\mathrm{d}}, \tag{2.11}$$

where both λ_{c} and λ_{d}, the CFL numbers for convection and diffusion respectively, are suitable positive constants independent of h and τ. A sufficient condition will be given in the next analysis process.

3. Preliminaries

In this section we first present some notations and norms which will be used throughout this paper, and then we give some properties of the finite element space and the LDG spatial discretizations.

3.1. Notations and norms

In this paper those norms and semi-norms in the Sobolev space are used as usual. For example, we denote by $L^2(D)$ those functions which are square integral in D. Let $(\cdot,\cdot)_D$ be the scalar inner product in $L^2(D)$, with the associated norm $\|\cdot\|_D$. For any integer $s \ge 0$, let $H^s(D)$ represent the space equipped with the norm $\|\cdot\|_{s,D}$, in which the function itself and the derivatives up to the s-th order are all in $L^2(D)$. If $D = \Omega = (a,b)$, we omit the subscript Ω for convenience.

Furthermore, we would like to consider the (mesh-dependent) broken Sobolev space

$$H^1(\mathcal{T}_h) = \Big\{\phi \in L^2(\Omega) : \phi|_{I_j} \in H^1(I_j),\, \forall j = 1,\dots,N\Big\}, \tag{3.1}$$

which contains the discontinuous finite element space V_h. Let $\Gamma = \{a,b\}$ be the boundary of domain Ω, and let Γ_h be the union of all element endpoints in the mesh. Denote by $\Gamma_h^{\rm int} = \Gamma_h \backslash \Gamma$ the union of all interior element endpoints. For any function $p \in H^1(\mathcal{T}_h)$, we define the following mesh-dependent notations

$$\|[\![p]\!]\|^2_{\Gamma_h^{\rm int}} = c\langle [\![p]\!], [\![p]\!]\rangle_h, \quad \|p\|^2_{\Gamma,\gamma} = c\left[(1+2\gamma_0)(p_0^+)^2 + (1+2\gamma_N)(p_N^-)^2\right], \tag{3.2a}$$

$$\|p\|^2_{\Gamma_h,\gamma} = \|[\![p]\!]\|^2_{\Gamma_h^{\rm int}} + \|p\|^2_{\Gamma,\gamma}. \tag{3.2b}$$

Note that the above notations all involve the convection velocity c, and the latter two also are related to the parameters γ_0 and γ_N.

Let $\{w^{n,\ell}\}_{\forall n}^{\ell=0,1,2}$ be a series of functions defined at every stage time levels. Following [21,22], we would like in this paper to adopt two series of simplifying notations

$$\mathbb{E}_1 w^n = w^{n,1} - w^n, \qquad \mathbb{D}_1 w^n = w^{n,1} - w^n, \tag{3.3a}$$

$$\mathbb{E}_2 w^n = 4w^{n,2} - w^{n,1} - 3w^n, \qquad \mathbb{D}_2 w^n = 2w^{n,2} - w^{n,1} - w^n, \tag{3.3b}$$

$$\mathbb{E}_3 w^n = \frac{3}{2}w^{n+1} - w^{n,2} - \frac{1}{2}w^n, \qquad \mathbb{D}_3 w^n = w^{n+1} - 2w^{n,2} + w^n. \tag{3.3c}$$

The left three notations describe the evolution of numerical solutions under the explicit TVDRK3 time-marching. The right three notations play similar roles as the time derivatives from the first order up to the third order, and have critical functions in our analysis. The detailed explanation will be given in (4.14).

It is worthy to point out that the notations in one column can be linearly expressed by those in the other column. For example, there hold the identities $\mathbb{E}_1 w^n = \mathbb{D}_1 w^n$ and

$$\mathbb{E}_2 w^n = 2\mathbb{D}_2 w^n + \mathbb{D}_1 w^n, \quad \mathbb{E}_3 w^n = \frac{3}{2}\mathbb{D}_3 w^n + \mathbb{D}_2 w^n + \mathbb{D}_1 w^n. \tag{3.4}$$

Consequently, there exists a positive constant C independent of n, ℓ and w, such that

$$\|\mathbb{E}_\ell w^n\| \le C \sum_{1\le\varsigma\le\ell} \|\mathbb{D}_\varsigma w^n\|, \quad \ell = 1,2,3. \tag{3.5}$$

3.2. Properties of the finite element space

Now we present some inverse properties with respect to V_h. For any function $v \in V_h$, there exists a positive constant $\mu > 0$ independent of v, h and j, such that

$$\|v_x\|_{I_j} \le \mu h^{-1} \|v\|_{I_j}, \tag{3.6a}$$

$$\|v\|_{\partial I_j} \le \mu^{1/2} h^{-1/2} \|v\|_{I_j}. \tag{3.6b}$$

Here $\|v\|_{\partial I_j} = ((v_{j-1}^+)^2 + (v_j^-)^2)^{1/2}$ is the L^2-norm on the boundary of I_j.

In this paper we will use two Gauss-Radau projections from $H^1(\mathcal{T}_h)$ to V_h, denoted by π_h^- and π_h^+, respectively. For the given function $p \in H^1(\mathcal{T}_h)$, the projection $\pi_h^{\pm} p$ is defined as the unique element in V_h such that, in each element $I_j = (x_{j-1}, x_j)$,

$$(\pi_h^- p - p, v)_{I_j} = 0 \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi_h^- p)_j^- = p_j^-, \tag{3.7a}$$

$$(\pi_h^+ p - p, v)_{I_j} = 0 \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi_h^+ p)_{j-1}^+ = p_{j-1}^+. \tag{3.7b}$$

They are different only at the exact collocation on different endpoint of each element, which provides a great help to obtain the optimal error estimates.

Denote by $\eta = p - \pi_h^{\pm} p$ the projection error. By a standard scaling argument [5], it is easy to obtain the following approximation property

$$\|\eta\|_{I_j} + h^{1/2}\|\eta\|_{\partial I_j} \le C\|p\|_{H^s(I_j)} h^{\min(k+1,s)}, \quad \forall j, \tag{3.8}$$

where the bounding constant $C > 0$ is independent of h and j.

In what follows we will mainly use the inverse inequalities and the approximation property in global form by summing up the above local inequalities over every $j = 1, 2, \ldots, N$. The conclusions are trivial and almost the same by dropping the subscripts, so omitted here.

3.3. Properties of the LDG spatial discretization

In this subsection we consider the LDG spatial discretizations. Let us start from the bilinear functionals associated with the inner information

$$B(\boldsymbol{w}, \boldsymbol{z}) = \mathcal{H}_{\rm int}(\boldsymbol{w}, v) + \mathcal{K}_{\rm int}(u, r), \tag{3.9}$$

where $\boldsymbol{w} = (u, q)$ and $\boldsymbol{z} = (v, r)$ are any functions in $H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h)$. Integrating on each cell I_j and then summing up for every $j = 1, 2, \ldots, N$, after a simple manipulation we will derive the following important identity

$$\begin{aligned} B(\boldsymbol{w}, \boldsymbol{z}) + B(\boldsymbol{z}, \boldsymbol{w}) = &- c\langle [\![u]\!], [\![v]\!]\rangle_h - c\left[(1+2\gamma_0)u_0^+ v_0^+ + (1+2\gamma_N)u_N^- v_N^-\right] \\ &+ \sqrt{d}\left[(1-\rho_N)(u_N^- r_N^- + v_N^- q_N^-) - \rho_0(u_0^+ r_0^+ + v_0^+ q_0^+)\right]. \end{aligned} \tag{3.10}$$

This process is straightforward, so omitted here. Similar results can be found in any literatures about the LDG methods, for example [10].

Based on this identity, we establish the following lemmas.

Lemma 3.1. *For any given positive constant ε, there holds the stability estimate*

$$B(\boldsymbol{w}, \boldsymbol{w}) \le -\frac{1}{2}\|u\|_{\Gamma_h,\gamma}^2 + \varepsilon\|u\|_{\Gamma,\gamma}^2 + \frac{1}{4\varepsilon}\mu P_{\rm e}^{-1} C_{\rho,\gamma}\|q\|^2, \tag{3.11}$$

for any $\boldsymbol{w} = (u, q) \in H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h)$, where $C_{\rho,\gamma}$ is a nonnegative constant depending on the parameters $\{\gamma, \rho\} := \{\gamma_0, \gamma_N, \rho_0, \rho_N\}$, in the form

$$C_{\rho,\gamma} = \frac{(1-\rho_N)^2}{1+2\gamma_N} + \frac{\rho_0^2}{1+2\gamma_0}. \tag{3.12}$$

Proof. The proof is straightforward. By taking $\boldsymbol{z} = \boldsymbol{w}$ in (3.10) we get

$$B(\boldsymbol{w}, \boldsymbol{w}) = -\frac{1}{2}\|u\|_{\Gamma_h,\gamma}^2 + \sqrt{d}\left[(1-\rho_N)u_N^- q_N^- - \rho_0 u_0^+ q_0^+\right].$$

Denote the last term on the right-hand side by Π. Then, an application of Cauchy-Schwartz inequality and Young's inequality yields

$$
\begin{aligned}
|\Pi| &\le \sqrt{\frac{d}{c}}\left(\frac{|1-\rho_N|}{\sqrt{1+2\gamma_N}}|q_N^-|\sqrt{c(1+2\gamma_N)}|u_N^-| + \frac{\rho_0}{\sqrt{1+2\gamma_0}}|q_0^+|\sqrt{c(1+2\gamma_0)}|u_0^+|\right)\\
&\le \sqrt{\frac{d}{c}}\left(\frac{(1-\rho_N)^2}{1+2\gamma_N}|q_N^-|^2 + \frac{\rho_0^2}{1+2\gamma_0}|q_0^+|^2\right)^{1/2}\|u\|_{\Gamma,\gamma}\\
&\le \sqrt{\frac{d\mu C_{\rho,\gamma}}{ch}}\|q\|\|u\|_{\Gamma,\gamma} \le \varepsilon\|u\|^2_{\Gamma,\gamma} + \frac{1}{4\varepsilon}\mu P_{\mathrm{e}}^{-1}C_{\rho,\gamma}\|q\|^2,
\end{aligned}
$$

where we have used inverse inequality (3.6b) on elements I_1 and I_N at the third step. This completes the proof of this lemma. □

Remark 3.1. From this lemma, the best parameters should be $\rho_0 = 0$ and $\rho_N = 1$, as the same as that in [7]. This done enables us to get rid of the affection of $C_{\rho,\gamma}$ and P_{e}, and leads to the biggest numerical stability. However, at this moment we do not make any assumption on these parameters. We are going to give a general estimate for arbitrary evaluation of these parameters, and then determine the parameters to obtain the optimal error estimate.

Lemma 3.2. *The bilinear functional $B(\boldsymbol{w},\boldsymbol{z})$ is continuous in the finite element space. Namely, for any $\boldsymbol{w}=(u,q)$ and $\boldsymbol{z}=(v,r)$ belonging to $V_h\times V_h$, there holds*

$$
|B(\boldsymbol{w},\boldsymbol{z})| \le \kappa_1 c\|u\|\|v\| + \kappa_2\sqrt{d}\|u\|\|r\| + \kappa_3\sqrt{d}\|q\|\|v\|. \tag{3.13}
$$

Let $\kappa = 1+\sqrt{2}$. The above bounding constants are given as

$$
\kappa_1 = \{\kappa + \max(\gamma_N,\gamma_0)\}\,\mu h^{-1},\quad \kappa_2 = \{\kappa + \max(\rho_0,|1-\rho_N|)\}\,\mu h^{-1},\quad \kappa_3 = \kappa\mu h^{-1}.
$$

Proof. This is a simple application of the inverse properties (3.6) and Cauchy-Schwartz inequality. More detailed proofs are given in the appendix.

Remark 3.2. Please keep in mind that the above constants κ_1,κ_2 and κ_3 are at least in $\mathcal{O}(h^{-1})$, in whatever case that they depend on the parameters $\{\gamma,\rho\}$ or not.

For the linear functionals associated with the boundary conditions, we have the following conclusion. To state it, we would like to introduce two notations

$$
\|\boldsymbol{g}\|_\gamma^2 = \frac{c(1+\gamma_0)^2}{1+2\gamma_0}g_a^2 + \frac{c\gamma_N^2}{1+2\gamma_N}g_b^2,\quad \|\boldsymbol{g}\|_\rho^2 = (1-\rho_0)^2g_a^2 + \rho_N^2 g_b^2, \tag{3.14}
$$

for the given boundary condition $\boldsymbol{g}=(g_a,g_b)$.

Lemma 3.3. *For any test function $(v,r)\in V_h\times V_h$, there holds*

$$
|\mathcal{H}_{\mathrm{bry}}(\boldsymbol{g};v)| \le \|\boldsymbol{g}\|_\gamma\|v\|_{\Gamma,\gamma}\,,\quad |\mathcal{K}_{\mathrm{bry}}(\boldsymbol{g};r)| \le \sqrt{\mu d h^{-1}}\|\boldsymbol{g}\|_\rho\|r\|. \tag{3.15}
$$

Proof. The proof is straightforward. From (2.9c) and (2.9d), an application of Cauchy-Schwartz inequality yields

$$
\begin{aligned}
|\mathcal{H}_{\mathrm{bry}}(\boldsymbol{g};v)| &\le \frac{\sqrt{c}(1+\gamma_0)|g_a|}{\sqrt{1+2\gamma_0}}\sqrt{c(1+2\gamma_0)}|v_0^+| + \frac{\sqrt{c}\gamma_N|g_b|}{\sqrt{1+2\gamma_N}}\sqrt{c(1+2\gamma_N)}|v_N^-|\\
&\le \|\boldsymbol{g}\|_\gamma\|v\|_{\Gamma,\gamma}\,,\\
|\mathcal{K}_{\mathrm{bry}}(\boldsymbol{g};r)| &\le \sqrt{d}\Big(|1-\rho_0||g_a||r_0^+| + \rho_N|g_b||r_N^-|\Big)\\
&\le \sqrt{d}\|\boldsymbol{g}\|_\rho\sqrt{|r_0^+|^2+|r_N^-|^2} \le \sqrt{\mu d h^{-1}}\|\boldsymbol{g}\|_\rho\|r\|,
\end{aligned}
$$

where we have used inverse property (3.6b) on the elements I_1 and I_N. Now we complete the proof of this lemma. □

4. Error Estimate

In this section we are ready to obtain the optimal error estimate by virtue of the energy analysis, along the same line as that in [21, 22]. The main difficulties in this paper come from the error estimate with respect to the intermediate boundary conditions and the diffusion term in equation (2.1). The analysis proceeds in several steps.

4.1. Error representation and the energy equation

Following [21], we introduce three reference functions, say, $\boldsymbol{W}^{(\ell)} = (U^{(\ell)}, Q^{(\ell)})$ for $\ell = 0, 1, 2$, associated with the TVDRK3 time discretization. In detail, let $\boldsymbol{W}^{(0)} = \boldsymbol{W}$ be the exact solution of problem (2.1) and then define

$$\boldsymbol{W}^{(1)} = \boldsymbol{W}^{(0)} + \tau D_t \boldsymbol{W}^{(0)}, \tag{4.1a}$$

$$\boldsymbol{W}^{(2)} = \frac{3}{4}\boldsymbol{W}^{(0)} + \frac{1}{4}\boldsymbol{W}^{(1)} + \frac{1}{4}\tau D_t \boldsymbol{W}^{(1)}. \tag{4.1b}$$

Here and below $D_t^i w$ denotes the i-th order time derivative of w; the superscript will be omitted if $i = 1$. For any indexes n and ℓ under consideration, the reference function at each stage time is defined by $\boldsymbol{W}^{n,\ell} = \boldsymbol{W}^{(\ell)}(x, t^n)$. Due to equation (2.1) and the above definitions, it is easy to see for any n and ℓ that

$$Q^{n,\ell} = \sqrt{d} U_x^{n,\ell}. \tag{4.2}$$

At each stage time, we denote the error between the exact (reference) solution and the numerical solution by $\mathbf{e}^{n,\ell} = (e_u^{n,\ell}, e_q^{n,\ell}) = (U^{n,\ell} - u^{n,\ell}, Q^{n,\ell} - q^{n,\ell})$. As the standard treatment in the finite element analysis, we would like to divide the stage error in the form $\mathbf{e} = \boldsymbol{\xi} - \boldsymbol{\eta}$, where

$$\boldsymbol{\eta} = (\eta_u, \eta_q) = (\pi_h^- U - U, \pi_h^+ Q - Q); \quad \boldsymbol{\xi} = (\xi_u, \xi_q) = (\pi_h^- U - u, \pi_h^+ Q - q), \tag{4.3}$$

here we have dropped the supper-scripts n and ℓ for simplicity. At this moment we would like to postpone the estimate to the projection error $\boldsymbol{\eta}$, and focus our sprits on how to estimate the error in the finite element space, say, $\boldsymbol{\xi} \in V_h \times V_h$.

To this end, we need to set up the error equations about $\boldsymbol{\xi}$. This process is based on the following lemma, when the exact solution is sufficiently smooth.

Lemma 4.1. *Assume the exact solution of problem* (2.1) *be sufficiently smooth. Then the following variational forms*

$$(\mathbb{E}_{\ell+1} U^n, v)_h = \tau \mathcal{H}(\boldsymbol{G}^{n,\ell}; \boldsymbol{W}^{n,\ell}, v) + (\zeta^{n,\ell}, v)_h, \tag{4.4a}$$

$$(Q^{n,\ell}, r)_h = \mathcal{K}(\boldsymbol{G}^{n,\ell}; U^{n,\ell}, r), \tag{4.4b}$$

hold for any function $(v, r) \in V_h \times V_h$. *Here* $\boldsymbol{G}^{n,\ell} = (U_a^{(\ell)}(t^n), U_b^{(\ell)}(t^n))$ *is called the reference boundary condition, since it is given in the same form as* (4.1).

Here $\zeta^{n,0} = \zeta^{n,1} = 0$ *is used only for notation's convenience, and* $\zeta^{n,2}$ *is the local truncation error in each step of TVDRK3 time-marching. Further, there exists a bounding constant* $C > 0$ *independent of* n, h *and* τ, *such that*

$$\|\zeta^{n,2}\| \le C\tau^{\frac{7}{2}} \|D_t^4 U\|_{L^2(t^n, t^{n+1}; L^2)}. \tag{4.5}$$

We will give the proof in the appendix and now continue to derive the error equations. Subtracting those variational forms in Lemma 4.1 from those in the LDGRK3 scheme (2.10), in the same order, we will obtain the following error equations

$$(\mathbb{E}_{\ell+1}\xi_u^n, v)_h = (\mathbb{E}_{\ell+1}\eta_u^n + \zeta^{n,\ell}, v)_h + \tau\mathcal{H}_{\rm bry}(\boldsymbol{\theta}^{n,\ell}; v) + \tau\mathcal{H}_{\rm int}(\boldsymbol{\xi}^{n,\ell}, v) - \tau\mathcal{H}_{\rm int}(\boldsymbol{\eta}^{n,\ell}, v), \tag{4.6a}$$
$$(\xi_q^{n,\ell}, r)_h = (\eta_q^{n,\ell}, r)_h + \mathcal{K}_{\rm bry}(\boldsymbol{\theta}^{n,\ell}; r) + \mathcal{K}_{\rm int}(\xi_u^{n,\ell}, r) - \mathcal{K}_{\rm int}(\eta_u^{n,\ell}, r), \tag{4.6b}$$

for any function $(v, r) \in V_h \times V_h$. Here $\boldsymbol{\theta}^{n,\ell} = \boldsymbol{G}^{n,\ell} - \boldsymbol{g}^{n,\ell} = (\theta_a^{n,\ell}, \theta_b^{n,\ell})$ reflects the error due to the boundary condition treatment.

For convenience, we would like to write error equation (4.6) in a compact form. Multiplying τ on (4.6b) and adding the results into (4.6a) yields the equivalent form

$$(\mathbb{E}_{\ell+1}\xi_u^n, v)_h + \tau(\xi_q^{n,\ell}, r)_h = \tau B(\boldsymbol{\xi}^{n,\ell}, \boldsymbol{z}) + \mathcal{Q}^{n,\ell}(\boldsymbol{z}), \tag{4.7}$$

where $\boldsymbol{z} = (v, r) \in V_h \times V_h$. The functional $\mathcal{Q}^{n,\ell}(\cdot)$ collects up the local residuals in the nth-step of TVDRK3 time-marching, defined in the form

$$\begin{aligned}\mathcal{Q}^{n,\ell}(\boldsymbol{z}) = {} & (\mathbb{E}_{\ell+1}\eta_u^n + \zeta^{n,\ell}, v)_h + \tau(\eta_q^{n,\ell}, r)_h - \tau B(\boldsymbol{\eta}^{n,\ell}, \boldsymbol{z}) \\ & + \tau\mathcal{H}_{\rm bry}(\boldsymbol{\theta}^{n,\ell}; v) + \tau\mathcal{K}_{\rm bry}(\boldsymbol{\theta}^{n,\ell}; r).\end{aligned} \tag{4.8}$$

These error equalities are fundamental and important in the energy analysis. In fact, we are able to make full use of these error equations to arrive at the energy equation.

Define three parameters $\beta_0 = \beta_1 = 1$ and $\beta_2 = 4$, throughout this paper. First we take the test functions $\boldsymbol{z} = \beta_\ell\boldsymbol{\xi}^{n,\ell}$ in (4.7) for $\ell = 0, 1, 2$, respectively. Then, sum up the above equalities. After a simple manipulation as that in [21], we can obtain the following energy evolution equation

$$\begin{aligned}3\|\xi_u^{n+1}\|^2 - 3\|\xi_u^n\|^2 = {} & -\sum_{\ell=0}^{2}\beta_\ell\|\xi_q^{n,\ell}\|^2\tau + \sum_{\ell=0}^{2}\beta_\ell B(\boldsymbol{\xi}^{n,\ell}, \boldsymbol{\xi}^{n,\ell})\tau + \sum_{\ell=0}^{2}\beta_\ell\mathcal{Q}^{n,\ell}(\boldsymbol{\xi}^{n,\ell}) \\ & + \|\mathbb{D}_2\xi_u^n\|^2 + 3(\mathbb{D}_3\xi_u^n, \mathbb{D}_1\xi_u^n)_h + 3(\mathbb{D}_3\xi_u^n, \mathbb{D}_2\xi_u^n)_h + 3\|\mathbb{D}_3\xi_u^n\|^2,\end{aligned} \tag{4.9}$$

where we have used relation (3.4) and expressed $\mathbb{E}_{\ell+1}\xi_u^n$ in terms of $\mathbb{D}_\ell\xi_u^n$. Each line on the right-hand side is denoted by $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$, respectively. In the next two subsections, we will devote to estimating them one by one.

For notation's convenience, in what follows we will use ε to represent a small positive constant, and use the notation C (or with subscript) to represent a generic positive constant which depends solely on the regularity of the exact solution and is independent of c, d, h, τ, n and ε. They may have different values in each occurrence.

4.2. Estimate to $\mathcal{R}_1^n$

In this subsection we are going to estimate $\mathcal{R}_1^n$, the first line on the right-hand side of energy equation (4.9). Each term involved there is denoted by $\mathcal{R}_{11}^n, \mathcal{R}_{12}^n$ and $\mathcal{R}_{13}^n$, one by one.

The first term $\mathcal{R}_{11}^n$ is a good term to ensure the numerical stability, and does not need any treatment. Hence we only estimate the last two terms. The second term $\mathcal{R}_{12}^n$ is easy to bound by using Lemma 3.1, in the form

$$\mathcal{R}_{12}^n \le \sum_{\ell=0}^{2}\beta_\ell\left\{-\frac{1}{2}\|\xi_u^{n,\ell}\|_{\Gamma_h,\gamma}^2 + \varepsilon\|\xi_u^{n,\ell}\|_{\Gamma,\gamma}^2 + \frac{\mu P_{\rm e}^{-1}C_{\rho,\gamma}}{4\varepsilon}\|\xi_q^{n,\ell}\|^2\right\}\tau. \tag{4.10}$$

The estimate to the last term $\mathcal{R}_{13}^n$ depends strongly on the properties of the residual functional $\mathcal{Q}^{n,\ell}(\cdot)$, which is stated in the next lemma in general form.

Lemma 4.2. *For any function $\boldsymbol{z}=(v,r)\in V_h\times V_h$, we have*

$$|\mathcal{Q}^{n,\ell}(\boldsymbol{z})|\le G_1^{n,\ell}\|v\|+G_2^{n,\ell}\|v\|_{\Gamma,\gamma}+G_3^{n,\ell}\|r\|, \tag{4.11}$$

where the bounding constants are given as following

$$G_1^{n,\ell}=\|\mathbb{E}_{\ell+1}\eta_u^n+\zeta^{n,\ell}\|, \tag{4.12a}$$

$$G_2^{n,\ell}=\|\boldsymbol{\theta}^{n,\ell}\|_\gamma\tau+\sqrt{\frac{c\gamma_0^2}{1+2\gamma_0}}|(\eta_u^{n,\ell})_0^+|\tau+\sqrt{\frac{d}{c(1+2\gamma_N)}}|(\eta_q^{n,\ell})_N^-|\tau, \tag{4.12b}$$

$$G_3^{n,\ell}=\sqrt{\mu dh^{-1}}\|\boldsymbol{\theta}^{n,\ell}\|_\rho\tau+\rho_0\sqrt{\mu dh^{-1}}|(\eta_u^{n,\ell})_0^+|\tau+\|\eta_q^{n,\ell}\|\tau. \tag{4.12c}$$

The proof is trivial by using the inverse property and Cauchy-Schwartz inequality, so we omit it here and put it in the appendix. Then a direct application of Lemma 4.2, together with Young's inequality, yield the estimate

$$\begin{aligned}|\mathcal{R}_{13}^n|&\le\sum_{\ell=0}^{2}\beta_\ell\left\{G_1^{n,\ell}\|\xi_u^{n,\ell}\|+G_2^{n,\ell}\|\xi_u^{n,\ell}\|_{\Gamma,\gamma}+G_3^{n,\ell}\|\xi_q^{n,\ell}\|\right\}\\&\le\varepsilon\tau\sum_{\ell=0}^{2}\beta_\ell\left(\|\xi_u^{n,\ell}\|^2+\|\xi_u^{n,\ell}\|_{\Gamma,\gamma}^2+\|\xi_q^{n,\ell}\|^2\right)+\frac{1}{4\varepsilon\tau}\sum_{\ell=0}^{2}\beta_\ell\sum_{\varsigma=1,2,3}|G_\varsigma^{n,\ell}|^2.\end{aligned} \tag{4.13}$$

Now we obtain the estimate to $\mathcal{R}_1^n$ by simple collection, and end this subsection.

4.3. Estimate to $\mathcal{R}_2^n$

In this subsection we will estimate $\mathcal{R}_2^n$, the second line on the right-hand side of energy equation (4.9). The analysis line strongly depend on the nice relations among four important functions $\mathbb{D}_\ell\boldsymbol{\xi}^n$ for $\ell=0,1,2,3$; here $\mathbb{D}_0\boldsymbol{\xi}^n=\boldsymbol{\xi}^n$. Noticing (3.4) and (4.7), after some manipulations we can yield the following variation forms

$$(\mathbb{D}_1\xi_u^n,v)_h+\tau(\mathbb{D}_0\xi_q^n,r)_h=\tau B(\mathbb{D}_0\boldsymbol{\xi}^n,\boldsymbol{z})+\mathbb{D}_0\mathcal{Q}^n(\boldsymbol{z}), \tag{4.14a}$$

$$(\mathbb{D}_2\xi_u^n,v)_h+\frac{\tau}{2}(\mathbb{D}_1\xi_q^n,r)_h=\frac{1}{2}\tau B(\mathbb{D}_1\boldsymbol{\xi}^n,\boldsymbol{z})+\frac{1}{2}\mathbb{D}_1\mathcal{Q}^n(\boldsymbol{z}), \tag{4.14b}$$

$$(\mathbb{D}_3\xi_u^n,v)_h+\frac{\tau}{3}(\mathbb{D}_2\xi_q^n,r)_h=\frac{1}{3}\tau B(\mathbb{D}_2\boldsymbol{\xi}^n,\boldsymbol{z})+\frac{1}{3}\mathbb{D}_2\mathcal{Q}^n(\boldsymbol{z}), \tag{4.14c}$$

for any test function $\boldsymbol{z}=(v,r)\in V_h\times V_h$. Here the linear functional $\mathbb{D}_\ell\mathcal{Q}^n(\cdot)$ is defined in form by replacing the function $w^{n,\ell}$ in (3.3) with the functional $\mathcal{Q}^{n,\ell}(\cdot)$.

Along the almost same line as that in Lemma 4.2, it is easy to get the conclusion about the residual functionals $\mathbb{D}_\ell\mathcal{Q}^n(\cdot)$ which is stated in the next lemma.

Lemma 4.3. *For any $\boldsymbol{z}=(v,r)$ in $V_h\times V_h$, we have that*

$$|\mathbb{D}_\ell\mathcal{Q}^n(\boldsymbol{z})|\le S_1^{n,\ell}\|v\|+S_2^{n,\ell}\|v\|_{\Gamma,\gamma}+S_3^{n,\ell}\|r\|,\quad \ell=0,1,2, \tag{4.15}$$

where the bounding constants are given as following

$$S_1^{n,\ell} = \|(\ell+1)\mathbb{D}_{\ell+1}\eta_u^n + \mathbb{D}_\ell\zeta^n\|, \tag{4.16a}$$

$$S_2^{n,\ell} = \|\mathbb{D}_\ell\boldsymbol{\theta}^n\|_\gamma\tau + \sqrt{\frac{c\gamma_0^2}{1+2\gamma_0}}|(\mathbb{D}_\ell\eta_u^n)_0^+|\tau + \sqrt{\frac{d}{c(1+2\gamma_N)}}|(\mathbb{D}_\ell\eta_q^n)_N^-|\tau, \tag{4.16b}$$

$$S_3^{n,\ell} = \sqrt{\mu d h^{-1}}\|\mathbb{D}_\ell\boldsymbol{\theta}^n\|_\rho\tau + \rho_0\sqrt{\mu d h^{-1}}|(\mathbb{D}_\ell\eta_u^n)_0^+|\tau + \|\mathbb{D}_\ell\eta_q^n\|\tau. \tag{4.16c}$$

Remark 4.1. Obviously $S_i^{n,0} = G_i^{n,0}$ for $i = 1, 2, 3$, since $\mathbb{D}_0\mathcal{Q}^n(\boldsymbol{z}) = \mathcal{Q}^n(\boldsymbol{z})$.

Based on variation forms (4.14) and Lemma 4.3, we can establish two series of important relations which are stated in the next lemmas.

Lemma 4.4. *There exists a bounding constant $K_1 > 0$ such that*

$$\|\mathbb{D}_{\ell+1}\xi_u^n\|^2 \le K_1\left\{c^2\kappa_1^2\tau^2\|\mathbb{D}_\ell\xi_u^n\|^2 + d\kappa_3^2\tau^2\|\mathbb{D}_\ell\xi_q^n\|^2 + |S_1^{n,\ell}|^2 + c\kappa_1|S_2^{n,\ell}|^2\right\}. \tag{4.17}$$

Lemma 4.5. *There exists a bounding constant $K_2 > 0$ such that*

$$\|\mathbb{D}_\ell\xi_q^n\|^2 \le K_2\left\{d\kappa_2^2\|\mathbb{D}_\ell\xi_u^n\|^2 + \tau^{-2}|S_3^{n,\ell}|^2\right\}. \tag{4.18}$$

The proofs are straightforward, so omitted here. However, we give them in the appendix for the completeness of this paper. It is worthy to mention that Lemma 4.4 is very helpful to control the error on the intermediate stage time by those on the integer time level, although in a little rough way.

Let us now come back to estimate $\mathcal{R}_2^n$, in which each including term is denoted by $\mathcal{R}_{2\varsigma}^n$ for $\varsigma = 1, 2, 3, 4$, one by one. The main technique used here is very similar as that in [21, 22].

As an important trick, we would like to estimate the sum of $\mathcal{R}_{21}^n$ and $\mathcal{R}_{22}^n$, but not estimate each term alone. By taking the test function $\boldsymbol{z} = 2\mathbb{D}_2\boldsymbol{\xi}^n$ in (4.14b) and $\boldsymbol{z} = 3\mathbb{D}_1\boldsymbol{\xi}^n$ in (4.14c), respectively, and combining them together, we can get the following identity

$$\begin{aligned}
\mathcal{R}_{21}^n + \mathcal{R}_{22}^n =& -\|\mathbb{D}_2\xi_u^n\|^2 + 2\|\mathbb{D}_2\xi_u^n\|^2 + 3(\mathbb{D}_3\xi_u^n, \mathbb{D}_1\xi_u^n)_h\\
=& -\|\mathbb{D}_2\xi_u^n\|^2 + \tau\left[B(\mathbb{D}_1\boldsymbol{\xi}^n, \mathbb{D}_2\boldsymbol{\xi}^n) + B(\mathbb{D}_2\boldsymbol{\xi}^n, \mathbb{D}_1\boldsymbol{\xi}^n)\right]\\
&+ \left[\mathbb{D}_1\mathcal{Q}^n(\mathbb{D}_2\boldsymbol{\xi}^n) + \mathbb{D}_2\mathcal{Q}^n(\mathbb{D}_1\boldsymbol{\xi}^n)\right] - 2\tau(\mathbb{D}_1\xi_q^n, \mathbb{D}_2\xi_q^n)_h\\
\equiv& -\|\mathbb{D}_2\xi_u^n\|^2 + V_1 + V_2 + V_3.
\end{aligned} \tag{4.19}$$

Here the first term, $-\|\mathbb{D}_2\xi_u^n\|^2$, provides the additional numerical stability due to the explicit TVDRK3 time-marching. By making full use of this fact, we can control those terms including notation $\mathbb{D}_2$ by this numerical stability under suitable temporal-spatial condition, with the help of inverse property (3.6) and Lemma 4.5.

Below we are going to estimate the last three terms on the right-hand side of (4.19), one by one. Noticing the anti-symmetric construction of V_1, we can get rid of the integration on each element, with the help of (3.10), and get the identity

$$\begin{aligned}
V_1 =& -c\langle[\![\mathbb{D}_2\xi_u^n]\!], [\![\mathbb{D}_1\xi_u^n]\!]\rangle_h\tau\\
&- c\left[(1+2\gamma_0)(\mathbb{D}_2\xi_u^n)_0^+(\mathbb{D}_1\xi_u^n)_0^+ + (1+2\gamma_N)(\mathbb{D}_2\xi_u^n)_N^-(\mathbb{D}_1\xi_u^n)_N^-\right]\tau\\
&+ \sqrt{d}\left[(1-\rho_N)(\mathbb{D}_1\xi_u^n)_N^-(\mathbb{D}_2\xi_q^n)_N^- - \rho_0(\mathbb{D}_1\xi_u^n)_0^+(\mathbb{D}_2\xi_q^n)_0^+\right]\tau\\
&+ \sqrt{d}\left[(1-\rho_N)(\mathbb{D}_2\xi_u^n)_N^-(\mathbb{D}_1\xi_q^n)_N^- - \rho_0(\mathbb{D}_2\xi_u^n)_0^+(\mathbb{D}_1\xi_q^n)_0^+\right]\tau.
\end{aligned}$$

Then we use Cauchy-Schwartz inequality to bound each above term. For example, the third line here is not greater than

$$\sqrt{d}\left[\frac{(1-\rho_N)^2}{c(1+2\gamma_N)}|(\mathbb{D}_2\xi_q^n)_N^-|^2+\frac{\rho_0^2}{c(1+2\gamma_0)}|(\mathbb{D}_2\xi_q^n)_0^+|^2\right]^{\frac{1}{2}}\|\mathbb{D}_1\xi_u^n\|_{\Gamma,\gamma}\tau\,,$$

thus is bounded by $\sqrt{\mu P_{\rm e}^{-1}C_{\rho,\gamma}}\|\mathbb{D}_2\xi_q^n\|\|\mathbb{D}_1\xi_u^n\|_{\Gamma,\gamma}\tau$, due to inverse property (3.6b) for $\mathbb{D}_2\xi_q^n$, and the definitions of $P_{\rm e}$ and $C_{\rho,\gamma}$. This process yields the final estimate

$$\begin{aligned}V_1\le{}&\tau\|[\![\mathbb{D}_2\xi_u^n]\!]\|_{\Gamma_h^{\rm int}}\|[\![\mathbb{D}_1\xi_u^n]\!]\|_{\Gamma_h^{\rm int}}+\tau\|\mathbb{D}_2\xi_u^n\|_{\Gamma,\gamma}\|\mathbb{D}_1\xi_u^n\|_{\Gamma,\gamma}\\&+\tau\sqrt{\mu P_{\rm e}^{-1}C_{\rho,\gamma}}\left[\|\mathbb{D}_1\xi_u^n\|_{\Gamma,\gamma}\|\mathbb{D}_2\xi_q^n\|+\|\mathbb{D}_2\xi_u^n\|_{\Gamma,\gamma}\|\mathbb{D}_1\xi_q^n\|\right].\end{aligned}$$

Keeping in mind that the additional numerical stability in (4.19), we are allowed to make the coefficients of those terms containing $\mathbb{D}_1$ small enough, without worrying about the coefficients of those terms involving $\mathbb{D}_2$. Hence, we apply Young's inequality to each above term and yield

$$\begin{aligned}V_1\le{}&\varepsilon\tau\|\mathbb{D}_1\xi_q^n\|^2+\varepsilon\tau\|[\![\mathbb{D}_1\xi_u^n]\!]\|_{\Gamma_h^{\rm int}}^2+2\varepsilon\tau\|\mathbb{D}_1\xi_u^n\|_{\Gamma,\gamma}^2+\frac{\tau}{4\varepsilon}\|[\![\mathbb{D}_2\xi_u^n]\!]\|_{\Gamma_h^{\rm int}}^2+\frac{\tau}{4\varepsilon}\|\mathbb{D}_2\xi_u^n\|_{\Gamma,\gamma}^2\\&+\frac{\mu P_{\rm e}^{-1}C_{\rho,\gamma}\tau}{4\varepsilon}\|\mathbb{D}_2\xi_q^n\|^2+\frac{\mu P_{\rm e}^{-1}C_{\rho,\gamma}\tau}{4\varepsilon}\|\mathbb{D}_2\xi_u^n\|_{\Gamma,\gamma}^2\,,\end{aligned}\tag{4.20}$$

where ε is a small positive constant. Similarly, we can estimate the rest terms by simple applications of Young's inequality, and get that

$$V_2\le\varepsilon\tau\sum_{\ell=1,2}\left\{\|\mathbb{D}_\ell\xi_u^n\|^2+\|\mathbb{D}_\ell\xi_u^n\|_{\Gamma,\gamma}^2+\|\mathbb{D}_\ell\xi_q^n\|^2\right\}+\frac{1}{4\varepsilon\tau}\sum_{\ell=1,2}\sum_{\varsigma=1,2,3}|S_\varsigma^{n,\ell}|^2,\tag{4.21}$$

$$V_3\le\varepsilon\tau\|\mathbb{D}_1\xi_q^n\|^2+\frac{1}{\varepsilon}\tau\|\mathbb{D}_2\xi_q^n\|^2,\tag{4.22}$$

where we have used Lemma 4.3 for (4.21). Till now we complete the estimate to every terms in the sum of $\mathcal{R}_{21}^n+\mathcal{R}_{22}^n$.

It is very easy to estimate the last two terms $\mathcal{R}_{23}^n$ and $\mathcal{R}_{24}^n$. By taking the test function $\boldsymbol{z}=3\mathbb{D}_2\boldsymbol{\xi}^n$ in (4.14c), we will get $\mathcal{R}_{23}^n=-\tau\|\mathbb{D}_2\xi_q^n\|^2+\tau B(\mathbb{D}_2\boldsymbol{\xi}^n,\mathbb{D}_2\boldsymbol{\xi}^n)+\mathbb{D}_2\mathcal{Q}^n(\mathbb{D}_2\boldsymbol{\xi}^n)$. Using Lemmas 3.1 and 4.3, as well as Young's inequality, we get the estimate

$$\begin{aligned}\mathcal{R}_{23}^n\le{}&-\tau\|\mathbb{D}_2\xi_q^n\|^2-\frac{\tau}{2}\|\mathbb{D}_2\xi_u^n\|_{\Gamma_h,\gamma}^2+2\varepsilon\tau(\|\mathbb{D}_2\xi_u^n\|^2+\|\mathbb{D}_2\xi_u^n\|_{\Gamma,\gamma}^2+\|\mathbb{D}_2\xi_q^n\|^2)\\&+\frac{\mu P_{\rm e}^{-1}C_{\rho,\gamma}\tau}{4\varepsilon}\|\mathbb{D}_2\xi_q^n\|^2+\frac{1}{4\varepsilon\tau}\sum_{\varsigma=1,2,3}|S_\varsigma^{n,2}|^2.\end{aligned}\tag{4.23}$$

Along the similar line, using Lemma 4.4 we can yield

$$\mathcal{R}_{24}^n\le 3K_1\left\{c^2\kappa_1^2\tau^2\|\mathbb{D}_2\xi_u^n\|^2+d\kappa_3^2\tau^2\|\mathbb{D}_2\xi_q^n\|^2+|S_1^{n,2}|^2+c\kappa_1|S_2^{n,2}|^2\right\}.\tag{4.24}$$

Now we can obtain the estimate to $\mathcal{R}_2^n$ by collecting the inequalities (4.19)–(4.24), and end this subsection.

4.4. A general estimate

In this subsection we are ready to get the error estimate at the successive time level. To do that, we first substitute the involving conclusions into (4.9). Then we leave alone those terms involving $\varepsilon^{-1}\|\mathbb{D}_2\xi_u^n\|_{\Gamma_h,\gamma}^2$ and $\varepsilon^{-1}\|\mathbb{D}_2\xi_q^n\|^2$, and amplify the other terms by virtue of triangular inequalities.

During this process, the main technique is how to deal with the two L^2-norm errors at the intermediate time stage $\|\xi_u^{n,1}\|$ and $\|\xi_u^{n,2}\|$ in (4.13). To do that, let us start from the simple inequalities

$$\|\xi_u^{n,1}\|^2 \le 2\|\xi_u^n\|^2 + 2\|\mathbb{D}_1\xi_u^n\|^2, \quad \|\xi_u^{n,2}\|^2 \le 3\|\xi_u^n\|^2 + \frac{3}{4}\|\mathbb{D}_1\xi_u^n\|^2 + \frac{3}{4}\|\mathbb{D}_2\xi_u^n\|^2. \tag{4.25}$$

Further, we bound $\|\mathbb{D}_1\xi_u^n\|^2$ by using Lemmas 4.4 and 4.5, in the form

$$\|\mathbb{D}_1\xi_u^n\|^2 \le K_1(c^2\kappa_1^2 + K_2 d^2\kappa_2^2\kappa_3^2)\|\xi_u^n\|^2\tau^2 + C\left\{|S_1^{n,0}|^2 + c\kappa_1|S_2^{n,0}|^2 + d\kappa_3^2|S_3^{n,0}|^2\right\}, \tag{4.26}$$

where the bounding constant $C > 0$ is independent of n, h and τ. Also we can bound the term $\|\mathbb{D}_2\xi_u^n\|^2$ along the similar line, however, we do nothing to this term in this paper.

After a complete collection, we will finally obtain the estimate

$$3\|\xi_u^{n+1}\|^2 - 3\|\xi_u^n\|^2 \le -\mathcal{S}^n + \mathcal{T}_1^n + \mathcal{T}_2^n + \mathcal{T}_3^n + \mathcal{T}_4^n + \mathcal{T}_5^n, \tag{4.27}$$

where $\mathcal{S}^n$ and $\mathcal{T}_i^n$ are used for different consideration.

In details, $\mathcal{S}^n$ represents the total numerical stability provided by the LDG spatial discretization and the TVDRK3 time-marching, in the form

$$\mathcal{S}^n = \frac{1}{2}\sum_{\ell=0}^{2}\beta_\ell\|\xi_u^{n,\ell}\|_{\Gamma_h,\gamma}^2\tau + \sum_{\ell=0}^{2}\beta_\ell\|\xi_q^{n,\ell}\|^2\tau + \|\mathbb{D}_2\xi_u^n\|^2. \tag{4.28a}$$

The next three terms in (4.27) reflect the so-called "anti-dissipation", in the form

$$\mathcal{T}_1^n = \sum_{\ell=0,1,2} C_0\varepsilon\|\xi_u^{n,\ell}\|_{\Gamma_h,\gamma}^2\tau + \sum_{\ell=0,1,2} C_0\left\{\varepsilon + \varepsilon^{-1}\mu P_{\mathrm{e}}^{-1}C_{\rho,\gamma} + d\kappa_3^2\tau\right\}\|\xi_q^{n,\ell}\|^2\tau, \tag{4.28b}$$

$$\mathcal{T}_2^n = C_0\varepsilon^{-1}(1 + \mu P_{\mathrm{e}}^{-1}C_{\rho,\gamma})\|\mathbb{D}_2\xi_q^n\|^2\tau, \tag{4.28c}$$

$$\mathcal{T}_3^n = C_0\varepsilon^{-1}(1 + \mu P_{\mathrm{e}}^{-1}C_{\rho,\gamma})\|\mathbb{D}_2\xi_u^n\|_{\Gamma_h,\gamma}^2\tau + C_0(\varepsilon\tau + c^2\kappa_1^2\tau^2)\|\mathbb{D}_2\xi_u^n\|^2, \tag{4.28d}$$

where ε is arbitrary small positive constant; assuming $0 < \varepsilon < 1$ hereafter. The next term $\mathcal{T}_4^n$ reflects the error accumulation, in the form

$$\mathcal{T}_4^n = C_0\varepsilon(1 + K_1c^2\kappa_1^2\tau^2 + K_1K_2d^2\kappa_2^2\kappa_3^2\tau^2)\|\xi_u^n\|^2\tau. \tag{4.28e}$$

Further, the last term $\mathcal{T}_5^n$ represents the local error in each TVDRK3 time-marching,

$$\begin{aligned}\mathcal{T}_5^n = {} & \frac{1}{4\varepsilon\tau}\sum_{\ell=0}^{2}\sum_{\varsigma=1,2,3}\beta_\ell|G_\varsigma^{n,\ell}|^2 + \frac{1}{2\varepsilon\tau}\sum_{\ell=0}^{2}\sum_{\varsigma=1,2,3}|S_\varsigma^{n,\ell}|^2 \\ & + C\left\{|S_1^{n,2}|^2 + c\kappa_1|S_2^{n,2}|^2\right\} + C\varepsilon\left\{|S_1^{n,0}|^2 + c\kappa_1|S_2^{n,0}|^2 + d\kappa_3^2|S_3^{n,0}|^2\right\}\tau.\end{aligned} \tag{4.28f}$$

Note that, all above bounding constants in (4.28), C_0 and C, are positive and independent of n, h, τ and ε. Here C_0 is used to emphasize a fixed constant that determines the temporal-spatial condition.

If the exact solution was taken as $U = Q = 0$, which implies $\boldsymbol{\eta} \equiv 0$, then the above error estimate (4.27) actually becomes the stability analysis. The statement will be given in the remark at the end of this section. Given this fact, it is a natural demand that the sum of $\mathcal{T}_1^n$, $\mathcal{T}_2^n$ and $\mathcal{T}_3^n$ should be controlled mainly by the numerical stability $\mathcal{S}^n$ under suitable temporal-spatial restrictions.

To this end, we firstly take the parameter ε small enough, for example, $\varepsilon = (4C_0)^{-1}$. Let us fix ε throughout this paper, and assume the following assumptions:

$$\textbf{(A1)} \quad \mu P_{\mathrm{e}}^{-1} C_{\rho,\gamma} \le (4C_0)^{-2}, \tag{4.29a}$$

$$\textbf{(A2)} \quad d\kappa_3^2\tau \le (4C_0)^{-1}, \quad \text{and} \quad d\kappa_2^2\tau \le K_2^{-1}(16C_0^2+1)^{-1}, \tag{4.29b}$$

$$\textbf{(A3)} \quad c\kappa_1\tau \le \frac{1}{2}\min\left((16C_0^2+1)^{-1}, 1/\sqrt{C_0}\right). \tag{4.29c}$$

The reasonability of above assumptions will be verified in the next subsection, if we have taken suitable parameters $\{\gamma,\rho\}$ and the time step τ under temporal-spatial restriction (2.11) with suitable CFL numbers λ_{c} and λ_{d}.

Due to assumptions **(A1)** and **(A2)**, it is easy to bound the term $\mathcal{T}_1^n$ by the first two terms in $\mathcal{S}^n$. Next, it follows from Lemma 4.5 and assumption **(A2)** that

$$\mathcal{T}_2^n \le \frac{K_2(16C_0^2+1)}{4}\left\{d\kappa_2^2\tau\|\mathbb{D}_2\xi_u^n\|^2 + \tau^{-1}|S_3^{n,2}|^2\right\} \le \frac{1}{4}\|\mathbb{D}_2\xi_u^n\|^2 + C\tau^{-1}|S_3^{n,2}|^2.$$

From inverse property (3.6b) and simple inequality $[\![v]\!]^2 \le 2(v^+)^2 + 2(v^-)^2$, it is followed that $\|v\|^2_{\Gamma_h,\gamma} \le \{2+2\max(\gamma_0,\gamma_N)\}\, c\mu h^{-1}\|v\|^2 \le 2c\kappa_1\|v\|^2$ for any $v \in V_h$. Thus we have

$$\mathcal{T}_3^n \le \left\{\frac{16C_0^2+1}{2}c\kappa_1\tau + \frac{\tau}{4} + C_0c^2\kappa_1^2\tau^2\right\}\|\mathbb{D}_2\xi_u^n\|^2 \le \frac{3}{4}\|\mathbb{D}_2\xi_u^n\|^2,$$

if $\tau \le 1$, due to assumption **(A3)**. Consequently, we have

$$\mathcal{T}_1^n + \mathcal{T}_2^n + \mathcal{T}_3^n \le \mathcal{S}^n + C\tau^{-1}|S_3^{n,2}|^2. \tag{4.30}$$

Further, it is easy to get that $\mathcal{T}_4^n \le C\|\xi_u^n\|^2\tau$ under the assumptions **(A2)** and **(A3)**. Till now the following inequality (4.27) is simplified in the form

$$3\|\xi_u^{n+1}\|^2 - 3\|\xi_u^n\|^2 \le C\|\xi_u^n\|^2\tau + C\mathcal{E}^n, \tag{4.31}$$

under the assumptions **(A1)**–**(A3)**, where

$$\mathcal{E}^n = \sum_{\ell=0,1,2}\sum_{\varsigma=1,2,3} Z_\varsigma^{n,\ell}, \tag{4.32}$$

is resulted from the expansion of $\mathcal{T}_5^n$ and the term $\tau^{-1}|S_3^{n,2}|^2$ in (4.30), with

$$Z_1^{n,\ell} = \tau^{-1}\|\mathbb{E}_{\ell+1}\eta_u^n + \zeta^{n,\ell}\|^2 + \|\eta_q^{n,\ell}\|^2\tau, \tag{4.33a}$$

$$Z_2^{n,\ell} = \left\{\frac{c\gamma_0^2}{1+2\gamma_0} + \frac{\rho_0^2\mu d}{h}\right\}|(\eta_u^{n,\ell})_0^+|^2\tau + \frac{d}{c(1+2\gamma_N)}|(\eta_q^{n,\ell})_N^-|^2\tau, \tag{4.33b}$$

$$Z_3^{n,\ell} = \|\boldsymbol{\theta}^{n,\ell}\|_\gamma^2\tau + \frac{\mu d}{h}\|\boldsymbol{\theta}^{n,\ell}\|_\rho^2\tau. \tag{4.33c}$$

Using the discrete Gronwall's inequality and triangle inequality, we obtain the general error estimate under the assumptions **(A1)**–**(A3)**, that

$$\|e_u^m\|^2 \le C\left(\|\xi_u^0\|^2 + \|\eta_u^m\|^2 + \sum_{n=0}^{m-1}\mathcal{E}^n\right) \le C\left(\|\eta_u^m\|^2 + \sum_{n=0}^{m-1}\mathcal{E}^n\right), \tag{4.34}$$

for arbitrary m satisfying $m\tau \le T$ since $\xi_u^0 = 0$, where the bounding constant $C>0$ depends solely on the final time T.

4.5. Setting of parameters and boundary treatment

In this subsection we want to get the optimal estimate to each term in the local truncation error $\mathcal{E}^n$, which is a sufficient condition to obtain the optimal error estimates from (4.34). During this process, we would like to determine the nice parameter setting and find out the nice treatment on the boundary condition.

We start our discussion from estimating the first including term $Z_1^{n,\ell}$. To this end, we would like to assume the exact solution of problem (2.1) is smooth enough, for example,

$$D_t^\ell U \in L^\infty(0,T;H^{k+2}), \quad (\ell=0,1,2), \quad \text{and} \quad D_t^4 U \in L^2(0,T;L^2). \tag{4.35}$$

Then it follows from (3.8) and the linearity of projections $\pi_h^\pm$ that the stage projection errors satisfy

$$\|\eta_u^{n,\ell}\| + \|\eta_q^{n,\ell}\| \le Ch^{k+1}, \qquad \|\mathbb{D}_{\ell+1}\eta_u^n\| + \|\mathbb{D}_{\ell+1}\eta_q^n\| \le Ch^{k+1}\tau, \tag{4.36}$$

where the bounding constant $C>0$ depends solely on the smoothness of the exact solution, independent of n, h and τ. Further, by noticing (4.5) we can assert that

$$Z_1^{n,\ell} \le Ch^{2k+2}\tau + C\|D_t^4 U\|^2_{L^2(t^n,t^{n+1};L^2)}\tau^6. \tag{4.37}$$

The second term $Z_2^{n,\ell}$ is related to the approximation error at the boundary from interior direction. Due to (3.8), there would exist a half order reduction in space for general smooth solution in Sobolev space. Therefore, in order to obtain the optimal error estimate, it is natural to take the involving coefficients in (4.33b) to be of the order at least $\mathcal{O}(h)$.

In this paper we would like to take the parameters independent of h, namely,

$$\gamma_0 = 0, \quad \rho_0 = 0, \tag{4.38a}$$

which make the coefficient of the first term to be zero. Given the symmetric property of this scheme (transforming the space variable x into $-x$), we know that $1-\rho_N$ plays the same function as ρ_0. Hence we would like to take $1-\rho_N=0$ due to (4.38a), namely,

$$\rho_N = 1. \tag{4.38b}$$

Since the parameter γ_N lies in the denominator in (4.33b), there is no way to make the coefficient of the second term to be zero unless $d=0$. However, by noticing the definition P_e, we would like to take $\gamma_N = \mathcal{O}(P_\mathrm{e}^{-1})$ to ensure that the coefficient of the second term is of the order $\mathcal{O}(h)$, for whatever case that the problem is convection-dominated or not. In this paper we just take

$$\gamma_N = P_\mathrm{e}^{-1}. \tag{4.38c}$$

As a consequence, we arrive at the estimate to the term $Z_2^{n,\ell}$ in the form

$$Z_2^{n,\ell} \le Ch^{2k+2}\tau, \tag{4.39}$$

where the bounding constant C is independent of n, h, τ and P_e. Similarly, noticing the definition of $\|\cdot\|_\gamma$ and $\|\cdot\|_\rho$, we have the estimate to $Z_3^{n,\ell}$ such as

$$Z_3^{n,\ell} \le C(1+P_\mathrm{e}^{-1}+cP_\mathrm{e}^{-1})\left\{(\theta_a^{n,\ell})^2 + (\theta_b^{n,\ell})^2\right\}\tau, \tag{4.40}$$

where the bounding constant C is independent of n, h, τ and P_e. The estimate (4.40) depends on the Péclet mesh number P_e and the detailed boundary treatment. For different cases, we have the following discussions.

Case 1: The considered problem is convection-dominated, say $P_{\rm e}^{-1} = \mathcal{O}(1)$. Noticing (4.40), we can obtain the optimal error estimate if we have coped with the boundary condition to ensure that $\theta_a^{n,\ell} = \theta_b^{n,\ell} = \mathcal{O}(\tau^3)$.

This aim is easy to implement in this case. The simplest consideration is to vanish those boundary errors at each intermediate time stage. To this end, we take the intermediate boundary condition as

$$\boldsymbol{g}^{n,\ell} = \boldsymbol{G}^{n,\ell}, \quad \ell = 0, 1, 2, \tag{4.41}$$

where $\boldsymbol{G}^{n,\ell}$ have been given in Lemma 4.1, with the same form as (4.1). In this paper we call this type of boundary treatment as "reference boundary condition".

Also there are other ways to satisfy the requirement. For example, the boundary condition is taken as the numerical solution of the explicit TVDRK3 time marching. In this paper we call this treatment as the Runge-Kutta boundary condtion. The detailed implementation is given in (5.2).

Case 2: The problem is not convection-dominated, say $P_{\rm e}^{-1} = \mathcal{O}(h^{-1})$. In order to obtain the optimal error estimate we have to take the boundary error smaller than that in **Case 1**. For instance, the boundary error must be not over than $\mathcal{O}(h^{1/2}\tau^3)$. Consequently, the reference boundary condition (4.41) still works well.

From the view point of (4.40), the Runge-Kutta boundary condition seems not good to obtain the optimal error estimate, since the numerical solution given by the explicit TVDRK3 time-marching (5.2) has the error of only $\mathcal{O}(\tau^3)$ at each time stage. But the numerical experiments shows it also works well; see Table 5.3 and Table 5.4 in section 5.

Given the above discussions, we would like to adopt in this paper the reference boundary condition (4.41) as a uniform method. Then it follows from inequalities (4.34) and (4.36) the optimal error estimate

$$\|e_u^n\|^2 \le C(h^{2k+2} + \tau^6), \tag{4.42}$$

for whatever case that the considered problem is convection-dominated or not, where the bounding constant C is independent of n, h, τ and the Péclet mesh number $P_{\rm e}$.

Before making the final conclusion, we have to verify the reasonability of assumptions **(A1)**–**(A3)**. Since the parameters $\{\gamma, \rho\}$ are taken as (4.38), there holds $C_{\rho,\gamma} = 0$, and thus assumption **(A1)** holds obviously. Moreover, $\kappa_1 = (\kappa + P_{\rm e}^{-1})\mu h^{-1}$ and $\kappa_2 = \kappa_3 = \kappa\mu h^{-1}$. As a consequence, assumptions **(A2)** and **(A3)** hold true if the time step τ satisfies the temporal-spatial restriction (2.11) with suitable CFL numbers, for example

$$\lambda_{\rm c} \le \frac{\alpha_{\rm c}}{2\kappa\mu}, \quad \lambda_{\rm d} \le \min\Big(\frac{\alpha_{\rm c}}{2\mu}, \frac{\alpha_{\rm d}}{(\kappa\mu)^2}\Big). \tag{4.43}$$

Here κ is defined in Lemma 3.2, μ is the inverse constant, and

$$\alpha_{\rm c} = \min\Big(\frac{1}{2\sqrt{C_0}}, \frac{1}{2(16C_0^2+1)}\Big), \quad \alpha_{\rm d} = \min\Big(\frac{1}{4C_0}, \frac{1}{K_2(16C_0^2+1)}\Big). \tag{4.44}$$

Till now we have obtained the optimal error estimate in L^2-norm, which is stated in the following theorem.

Theorem 4.1. *Let u be the numerical solution of LDGRK3 method* (2.10) *with the parameter setting* (4.38) *and the reference boundary condition* (4.41). *The finite element space V_h is the piecewise polynomials with degree $k \ge 1$ on the regular triangulations of $\Omega = (a, b)$, and the time step τ satisfies* (2.11) *with the suitable CFL numbers* (4.43).

Let U be the exact solution of problem (2.1) *and satisfy the smoothness assumption* (4.35)*, then there holds the following error estimate*

$$\max_{n\tau\le T}\|U(t^n)-u^n\|\le C(h^{k+1}+\tau^3), \tag{4.45}$$

where the bounding constant $C>0$ is independent of h and τ.

Before ending this section, we would like to give a remark on the numerical stability for the LDGRK3 method (2.10). As we have mentioned before, by taking $U=0$, along the same line we can derive the following stability result

$$\|u^m\|^2\le C\Big[\|u^0\|^2+\sum_{n=0}^{m-1}\sum_{\ell=0,1,2}\Big(\|\boldsymbol{g}^{n,\ell}\|_\gamma^2+\frac{\mu d}{h}\|\boldsymbol{g}^{n,\ell}\|_\rho^2\Big)\tau\Big], \tag{4.46}$$

for arbitrary m satisfying $m\tau\le T$, where the bounding constant $C>0$ is independent of n,h and τ; maybe depend on the final time T. Noting the factor h^{-1} in the right-hand side, this conclusion is not good in the viewpoint of stability. However, it works well to achieve our purpose in the optimal error estimate.

5. Numerical Experiments

The purpose of this section is to validate numerically the optimal error estimate, and investigate the effective of the reference boundary condition at the same time.

In all numerical tests we will compare with three types of boundary condition treatments. The first one is the exact boundary condition, which is the natural treatment at the first glance. It is equal to the given boundary condition at the stage time level of TVDRK3 time-marching. Namely, denote $\mathbf{U}_{\rm bry}(t)=(U_a(t),U_b(t))$, and define

$$\boldsymbol{g}_{\rm ex}^n=\mathbf{U}_{\rm bry}(t^n),\quad \boldsymbol{g}_{\rm ex}^{n,1}=\mathbf{U}_{\rm bry}(t^n+\tau),\quad \boldsymbol{g}_{\rm ex}^{n,2}=\mathbf{U}_{\rm bry}(t^n+\frac{\tau}{2}). \tag{5.1}$$

The second one is the reference boundary condition (4.41), as we have discussed in this paper. The last one is the Runge-Kutta boundary condition, denoted by $\boldsymbol{g}_{\rm RK}^{n,\ell}$. It is equal to the approximation solution for the ordinary differential equation with respect to $\boldsymbol{g}$,

$$\boldsymbol{g}'(t)=\mathbf{U}'_{\rm bry}(t),\ t>0,\quad \boldsymbol{g}(0)=\mathbf{U}_{\rm bry}(0), \tag{5.2}$$

by using the explicit TVDRK3 time-marching. Similar treatment has been used in [4,22]. The reference boundary condition discussed in this paper can be looked upon as the local treatment of the third method.

We will adopt the LDGRK3 scheme to solve this problem on both uniform and nonuniform meshes. The uniform mesh has the mesh size $h=(b-a)/N$, where N is the number of used cells. The nonuniform mesh is given by randomly perturbing each node in the uniform mesh by up 10%. Denote $h_{\min}=\min_{1\le j\le N}h_j$. In the LDGRK3 method, we take $\gamma_0=\rho_0=0$, $\gamma_N=P_{\rm e}^{-1}$ and $\rho_N=1$. The time step is chosen as

$$\tau=\min(\lambda_{\rm c}h_{\min}/c,\lambda_{\rm d}h_{\min}^2/d), \tag{5.3}$$

with the CFL numbers $\lambda_{\rm c}$ and $\lambda_{\rm d}$ which will be given in each computation.

Table 5.1: errors and convergence orders for LDGRK3 method with three different boundary treatments on uniform meshes. Here $c = 1$, $d = 10^{-8}$ and $k = 2$.

	N	L^∞- error	L^∞- order	L^2- error	L^2- order
exact b.c.	10	3.7540E-05		5.8993E-06	
	20	9.7366E-06	1.9470	9.3152E-07	2.6629
	40	2.0560E-06	2.2436	1.3316E-07	2.8064
	80	4.1970E-07	2.2924	1.8771E-08	2.8266
	160	1.1722E-07	1.8402	3.4284E-09	2.4530
	320	2.5382E-08	2.2073	5.1301E-10	2.7405
rk b.c.	10	1.6166E-05		4.7938E-06	
	20	2.0221E-06	2.9990	5.9863E-07	3.0014
	40	2.5280E-07	2.9998	7.4845E-08	2.9997
	80	3.1602E-08	2.9999	9.3565E-09	2.9999
	160	3.9504E-09	3.0000	1.1695E-09	3.0001
	320	4.9625E-10	2.9929	1.4600E-10	3.0019
reference b.c.	10	1.6652E-05		4.7751E-06	
	20	2.0828E-06	2.9991	5.9657E-07	3.0008
	40	2.6039E-07	2.9998	7.4556E-08	3.0003
	80	3.2551E-08	2.9999	9.3186E-09	3.0001
	160	4.0690E-09	3.0000	1.1648E-09	3.0001
	320	5.0954E-10	2.9974	1.4560E-10	2.9999

Let $U(x,t) = e^{-dt}\sin(x - ct)$ be the exact solution of problem (2.1), in the interval $(a,b) = (0,1)$. The initial solution and Dirichlet boundary condition are given by this exact solution. We will verify our error estimate for whatever case the problem is convection-dominated or not.

We firstly consider a convection-dominated case, namely, $c = 1$ and $d = 10^{-8}$. In this case, we use piecewise quadratic polynomials ($k = 2$), and compute till the final time $T = 10$. The time step is determined by (5.3) with CFL numbers $\lambda_{\rm c} = 0.18$ and $\lambda_{\rm d} = 0.01$. Note that $\tau = \mathcal{O}(h)$, since d is too small. The errors and convergence orders in L^∞-norm and L^2-norm, under different boundary condition treatments, are listed in Table 5.1 and Table 5.2, for the uniform mesh and the nonuniform mesh, respectively. The convergence order in this paper is computed by

$$\text{order}_N = \frac{\log(\text{error}_{\frac{N}{2}}) - \log(\text{error}_N)}{\log(h_{\frac{N}{2}}) - \log(h_N)}, \tag{5.4}$$

where error_N and h_N represent respectively the error (in L^∞-norm or L^2-norm) at the final time T and the maximum cell size when the used mesh number is equal to N.

Then we consider the problem (2.1) with $c = d = 0.1$, which is not convection-dominated. In this case, we use piecewise polynomials with degree $k = 5$, and compute till the final time $T = 0.1$. The time step is determined by (5.3) with the CFL numbers $\lambda_{\rm c} = 0.05$ and $\lambda_{\rm d} = 0.001$. In this case $\tau = \mathcal{O}(h^2)$. The errors and convergence orders in L^∞-norm and L^2-norm, under different boundary condition treatments, are listed in Table 5.3 and Table 5.4, for the uniform mesh and the nonuniform mesh, respectively. The convergence order is also given by (5.4).

We can see from four tables that, the convergence orders drop seriously if we take the exact boundary condition (5.1) in both cases, on either uniform or nonuniform meshes. While when we take the reference boundary condition (4.41), we can observe the optimal error order

Table 5.2: errors and convergence orders for LDGRK3 method with different boundary treatments on nonuniform meshes (10%). Here $c = 1$, $d = 10^{-8}$ and $k = 2$.

	N	$h_{\max}$	$h_{\min}$	L^∞- error	L^∞- order	L^2- error	L^2- order
exact b.c.	10	0.1144	0.0864	3.7551E-05		6.1522E-06	
	20	0.0575	0.0453	6.1040E-06	2.6438	7.5830E-07	3.0466
	40	0.0295	0.0206	1.1966E-06	2.4387	1.0034E-07	3.0268
	80	0.0146	0.0107	2.5144E-07	2.2094	1.3671E-08	2.8231
	160	0.0074	0.0053	6.6970E-08	1.9567	2.1120E-09	2.7623
	320	0.0037	0.0025	1.3540E-08	2.3026	3.0277E-10	2.7978
rk b.c.	10	0.1139	0.0876	2.2135E-05		5.0125E-06	
	20	0.0580	0.0428	3.1130E-06	2.9067	6.3791E-07	3.0547
	40	0.0285	0.0212	3.7857E-07	2.9624	7.8833E-08	2.9399
	80	0.0146	0.0103	5.0881E-08	3.0105	1.0025E-08	3.0935
	160	0.0074	0.0052	6.5816E-09	3.0272	1.2539E-09	3.0770
	320	0.0037	0.0025	8.4695E-10	2.9546	1.5641E-10	2.9995
reference b.c.	10	0.1061	0.0935	1.8904E-05		4.8622E-06	
	20	0.0592	0.0417	3.2979E-06	2.9919	6.3072E-07	3.4997
	40	0.0291	0.0212	3.9673E-07	2.9830	8.0280E-08	2.9035
	80	0.0147	0.0106	4.9945E-08	3.0407	9.9044E-09	3.0703
	160	0.0074	0.0050	6.6016E-09	2.9331	1.2526E-09	2.9971
	320	0.0037	0.0025	8.3870E-10	3.0201	1.5552E-10	3.0538

Table 5.3: errors and convergence orders for LDGRK3 method with different boundary treatments on uniform meshes. Here $c = d = 0.1$ and $k = 5$.

	N	L^∞-error	L^∞-order	L^2-error	L^2-order
exact b.c.	10	1.4273E-11		1.1144E-12	
	20	8.9531E-13	3.9948	4.7547E-14	4.5508
	40	5.5906E-14	4.0013	2.0929E-15	4.5058
	80	3.4915E-15	4.0011	9.2448E-17	4.5007
	160	2.1813E-16	4.0006	4.0850E-18	4.5002
	320	1.3631E-17	4.0003	1.8052E-19	4.5001
rk b.c.	10	2.2384E-12		3.5532E-13	
	20	3.6918E-14	5.9220	5.6248E-15	5.9812
	40	5.9214E-16	5.9623	8.8213E-17	5.9947
	80	9.3721E-18	5.9814	1.3798E-18	5.9985
	160	1.4741E-19	5.9905	2.1569E-20	5.9993
	320	2.3435E-21	5.9751	3.3984E-22	5.9880
reference b.c.	10	2.2383E-12		3.5532E-13	
	20	3.6918E-14	5.9220	5.6248E-15	5.9812
	40	5.9213E-16	5.9623	8.8213E-17	5.9947
	80	9.3719E-18	5.9814	1.3798E-18	5.9985
	160	1.4741E-19	5.9905	2.1569E-20	5.9993
	320	2.3434E-21	5.9750	3.3984E-22	5.9880

Table 5.4: errors and convergence orders for LDGRK3 method with different boundary treatments on nonuniform meshes (10%). Here $c = 0.1$, $d = 0.1$ and $k = 5$.

	N	$h_{\max}$	$h_{\min}$	L^∞- error	L^∞- order	L^2- error	L^2- order
exact b.c.	10	0.1100	0.0914	5.2961E-12		5.7904E-13	
	20	0.0577	0.0435	2.4842E-13	4.7418	1.5987E-14	5.5632
	40	0.0288	0.0213	1.3864E-14	4.1634	5.6041E-16	4.8343
	80	0.0144	0.0103	7.8514E-16	4.1422	2.4317E-17	4.5264
	160	0.0072	0.0052	4.4945E-17	4.1267	9.0798E-19	4.7432
	320	0.0037	0.0026	2.9339E-18	4.0993	4.2871E-20	4.5859
rk b.c.	10	0.1100	0.0914	3.2615E-12		4.0124E-13	
	20	0.0577	0.0435	7.2242E-14	5.9047	7.2305E-15	6.2244
	40	0.0288	0.0213	1.0096E-15	6.1610	1.0988E-16	6.0401
	80	0.0144	0.0103	1.7120E-17	5.8820	1.6457E-18	6.0611
	160	0.0072	0.0052	2.7979E-19	5.9352	2.6090E-20	5.9785
	320	0.0037	0.0026	5.8409E-21	5.8117	4.2248E-22	6.1933
reference b.c.	10	0.1100	0.0914	3.2615E-12		4.0124E-13	
	20	0.0577	0.0435	7.2242E-14	5.9047	7.2305E-15	6.2244
	40	0.0288	0.0213	1.0096E-15	6.1610	1.0988E-16	6.0401
	80	0.0144	0.0103	1.7120E-17	5.8820	1.6457E-18	6.0611
	160	0.0072	0.0052	2.7979E-19	5.9352	2.6090E-20	5.9785
	320	0.0037	0.0026	5.8409E-21	5.8117	4.2248E-22	6.1933

without the reduction of accuracy. This verifies that (4.41) is a good way to deal with Dirichlet boundary condition uniformly for convection-diffusion equation and ensure the optimal-order accuracy, as we stated in Theorem 4.1.

An interesting fact is that (5.2) still provides satisfying simulations, although we do not prove this setting in this paper. In fact, the reference boundary condition (4.41) is just a local implementation of Runge-Kutta boundary condition (5.2), by resetting the initial solution at each step as the exact solution.

6. Concluding Remarks

In this paper we discuss a fully-discrete LDGRK3 scheme for the convection-diffusion problems with Dirichlet boundary conditions, where the numerical solution is updated by the explicit TVDRK3 algorithm. We adopt energy technique and obtain the optimal error estimate in L^2-norm. In the analysis process, we re-establish the special numerical flux similar as that in [7], and present a good setting for boundary conditions at each intermediate stage time. This special and simple treatment is suitable for whatever case that the problem is convection-dominant or not. In the further work, we will develop the above analysis to other boundary conditions and to the multidimensional problems. Also we will consider the nonlinear convection-diffusion equations.

7. Appendix

Proof of Lemma 3.2. We only need to use the Cauchy-Schwartz inequality and inverse properties (3.6) to prove

$$|\mathcal{H}_{\rm int}(\boldsymbol{w},v)| \le \Big\{1+\sqrt{2}+\max(\gamma_N,\gamma_0)\Big\} c\mu h^{-1}\|u\|\|v\| + \sqrt{d}(1+\sqrt{2})\mu h^{-1}\|q\|\|v\|, \tag{7.1a}$$

$$|\mathcal{K}_{\rm int}(u,r)| \le \sqrt{d}\Big\{1+\sqrt{2}+\max(\rho_0,|1-\rho_N|)\Big\}\mu h^{-1}\|u\|\|r\|. \tag{7.1b}$$

We take (7.1a) as an example. By definition (2.9a), we can get

$$\begin{aligned}|\mathcal{H}_{\rm int}(\boldsymbol{w},v)| \le{}& |(cu,v_x)_h| + |\langle cu^-,[\![v]\!]\rangle_h| + |c(1+\gamma_N)u_N^-v_N^-| + |c\gamma_0 u_0^+v_0^+|\\ &+|(\sqrt{d}q,v_x)_h| + |\langle\sqrt{d}q^+,[\![v]\!]\rangle_h| + |\sqrt{d}(q_N^-v_N^- - q_0^+v_0^+)|.\end{aligned}$$

As a typical term, the first line on the right-hand side is bounded as follows:

$$\begin{aligned}&|(cu,v_x)_h| \le c\|u\|\|v_x\| \le c\mu h^{-1}\|u\|\|v\|,\\ &|\langle cu^-,[\![v]\!]\rangle_h| + |cu_N^-v_N^-| \le c\Big(\sum_{j=1}^{N-1}|u_j^-|^2+|u_N^-|^2\Big)^{\frac12}\Big(2\sum_{j=1}^{N-1}((v_j^-)^2+(v_j^+)^2)+(v_N^-)^2\Big)^{\frac12}\\ &\qquad\qquad\qquad\qquad\qquad\le \sqrt{2}c\mu h^{-1}\|u\|\|v\|,\\ &|c\gamma_N u_N^-v_N^-| + |c\gamma_0u_0^+v_0^+| \le c\max(\gamma_N,\gamma_0)|((u_N^-)^2+(u_0^+)^2)^{\frac12}((v_N^-)^2+(v_0^+)^2)^{\frac12}\\ &\qquad\qquad\qquad\qquad\qquad\le \max(\gamma_N,\gamma_0)c\mu h^{-1}\|u\|\|v\|.\end{aligned}$$

Similar estimate holds for the second line. Then we get (7.1a). It completes the proof of this lemma. □

Proof of Lemma 4.1. Let us start from the proof for equality (4.4a) for $\ell=2$. Here and below we would like to drop the arguments (x,t) for notation's simplicity. By the definition of the reference functions of $U^{(0)},U^{(1)}$ and $U^{(2)}$, we derive

$$\frac13U^{(0)}+\frac23U^{(2)}+\frac23\tau D_tU^{(2)} = U^{(0)}+\tau D_tU^{(0)}+\frac{\tau^2}{2}D_t^2U^{(0)}+\frac{\tau^3}{6}D_t^3U^{(0)}, \tag{7.2}$$

by simple manipulation. By virtue of the Taylor's expansion in time, hence we get

$$U(x,t+\tau) = \frac13U^{(0)}+\frac23U^{(2)}+\frac23\tau D_tU^{(2)}+\frac16\int_t^{t+\tau}(t+\tau-s)^3D_t^4U(x,s)\,{\rm d}s, \tag{7.3}$$

where the last term on the right-hand side is denoted by $\zeta(x,t)$. By Cauchy-Schwartz inequality and Fubini's Theorem, we have the estimate

$$\begin{aligned}\|\zeta(x,t)\|^2 &= \int_I\left(\frac16\int_t^{t+\tau}(t+\tau-s)^3D_t^4U(x,s)\,{\rm d}s\right)^2{\rm d}x\\ &\le \frac{\tau^7}{252}\int_I\int_t^{t+\tau}|D_t^4U(x,s)|^2\,{\rm d}s{\rm d}x\\ &\le \frac{\tau^7}{252}\int_t^{t+\tau}\int_I|D_t^4U(x,s)|^2\,{\rm d}x{\rm d}s \le C\|D_t^4U\|^2_{L^2(t,t+\tau;L^2)}\tau^7,\end{aligned} \tag{7.4}$$

since $D_t^4U\in L^2(0,T;L^2)$. Let $t=t^n$ and denote $\zeta^{n,2}=\zeta(x,t^n)$. This yields (4.5).

Then we take time $t = t^n$ in (7.3) and multiply the test function v on both sides of this equality. A simple integration by parts yields the first equality in (4.4) for $\ell = 2$, due to the sufficient regularity of the considered model problem (2.1).

Other equalities can be obtained by the considered convection-diffusion problem (2.1) and the definitions of reference functions, along the same line as above. This completes the proof of this lemma. $\square$

Proof of Lemma 4.2. This lemma can be proved by noticing the identities

$$B(\boldsymbol{\eta}, \boldsymbol{z}) = \mathcal{H}_{\rm int}(\boldsymbol{\eta}, v) + \mathcal{K}_{\rm int}(\eta_u, r) = \sqrt{d}\eta_{q,N}^- v_N^- - \gamma_0 c\eta_{u,0}^+ v_0^+ - \rho_0\sqrt{d}\eta_{u,0}^+ r_0^+, \tag{7.5}$$

by the property of the projection error presented in subsection 3.2. Here and below we drop the script n, ℓ for simplicity. This gives us that

$$\begin{aligned}
|B(\boldsymbol{\eta}, \boldsymbol{z})| &\le \frac{\sqrt{d}|\eta_{q,N}^-|}{\sqrt{c(1+2\gamma_N)}}\sqrt{c(1+2\gamma_N)}|v_N^-| + \frac{\sqrt{c}\gamma_0|\eta_{u,0}^+|}{\sqrt{(1+2\gamma_0)}}\sqrt{c(1+2\gamma_0)}|v_0^+| + \sqrt{d}\rho_0|\eta_{u,0}^+||r_0^+| \\
&\le \left[\sqrt{\frac{c\gamma_0^2}{1+2\gamma_0}}|\eta_{u,0}^+| + \sqrt{\frac{d}{c(1+2\gamma_N)}}|\eta_{q,N}^-|\right]\|v\|_{\Gamma,\gamma} + \sqrt{\mu d h^{-1}}\rho_0|\eta_{u,0}^+|\|r\|,
\end{aligned}$$

where we have used the inverse property (3.6b) at the last step. By Lemma 3.3

$$|\mathcal{H}_{\rm bry}(\boldsymbol{\theta}; v) + \mathcal{K}_{\rm bry}(\boldsymbol{\theta}; r)| \le \|\boldsymbol{\theta}\|_\gamma\|v\|_{\Gamma,\gamma} + \sqrt{\mu d h^{-1}}\|\boldsymbol{\theta}\|_\rho\|r\|.$$

And by Cauchy-Schwartz inequality we can easily obtain

$$|(\mathbb{E}_{\ell+1}\eta_u^n + \zeta^{n,\ell}, v)_h + \tau(\eta_q^{n,\ell}, r)_h| \le \|\mathbb{E}_{\ell+1}\eta_u^n + \zeta^{n,\ell}\|\|v\| + \tau\|\eta_q^{n,\ell}\|\|r\|.$$

Finally, we can prove this lemma by substituting the above results into the expression of $\mathcal{Q}(\cdot)$.

Proof of Lemma 4.4. In this proof we would like to drop the script n for simplicity. By taking the test function $\boldsymbol{z} = (\mathbb{D}_{\ell+1}\xi_u, 0)$ in (4.14), we have, for $\ell = 0, 1, 2$, that

$$\begin{aligned}
&(\ell+1)\|\mathbb{D}_{\ell+1}\xi_u\|^2 = \tau B(\mathbb{D}_\ell\boldsymbol{\xi}, \boldsymbol{z}) + \mathbb{D}_\ell\mathcal{Q}(\boldsymbol{z}) \\
\le &\left\{\kappa_1 c\tau\|\mathbb{D}_\ell\xi_u\| + \kappa_3\sqrt{d}\tau\|\mathbb{D}_\ell\xi_q\| + S_1^{n,\ell}\right\}\|\mathbb{D}_{\ell+1}\xi_u\| + S_2^{n,\ell}\|\mathbb{D}_{\ell+1}\xi_u\|_{\Gamma,\gamma},
\end{aligned} \tag{7.6}$$

where we have used Lemmas 3.2 and 4.3. For the last term on the right-hand side, we use the inverse property

$$\|p\|_{\Gamma,\gamma} \le \sqrt{c\mu_\gamma h^{-1}}\|p\|, \quad \forall p \in V_h, \tag{7.7}$$

due to the definition of norm $\|\cdot\|_{\Gamma,\gamma}$ and the inverse property (3.6b), and then get

$$S_2^{n,\ell}\|\mathbb{D}_{\ell+1}\xi_u\|_{\Gamma,\gamma} \le \varepsilon\|\mathbb{D}_{\ell+1}\xi_u\|^2 + \frac{1}{4\varepsilon}c\mu_\gamma h^{-1}|S_2^{n,\ell}|^2, \tag{7.8}$$

for arbitrary positive ε, where $\mu_\gamma = \max(1+2\gamma_0, 1+2\gamma_N)\mu \le C\kappa_1 h$. The parameter κ_1 has been defined in Lemma 3.2.

Using the Young's inequality again for the first term on the right-hand side of (7.6) and letting ε small enough, we can complete the proof of this lemma. $\square$

Proof of Lemma 4.5. As the above proof, in the below we will drop the script n for simplicity. By taking the test function $\boldsymbol{z} = (0, \mathbb{D}_\ell \xi_q)$ in (4.14), we get

$$\begin{aligned}
\|\mathbb{D}_\ell \xi_q\|^2 &= B(\mathbb{D}_\ell \boldsymbol{\xi}, \boldsymbol{z}) + \frac{1}{\tau}\mathbb{D}_\ell \mathcal{Q}(\boldsymbol{z}) \\
&\leq \kappa_2 \sqrt{d} \|\mathbb{D}_\ell \xi_u\| \|\mathbb{D}_\ell \xi_q\| + \frac{1}{\tau}|S_3^{n,\ell}| \|\mathbb{D}_\ell \xi_q\| \\
&\leq 2\varepsilon \|\mathbb{D}_\ell \xi_q\|^2 + \frac{1}{4\varepsilon} d\kappa_2^2 \|\mathbb{D}_\ell \xi_u\|^2 + \frac{1}{4\varepsilon\tau^2}|S_3^{n,\ell}|^2, \qquad (7.9)
\end{aligned}$$

owing to Lemmas 3.2 and 4.3. Finally, by choosing ε property, we get the desired result (4.18) in this lemma. □

Acknowledgments. The research is supported by NSFC grants 10871093 and 11271187, and partially by NSFC grant 10931004.

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