Journal of Computational Mathematics
Vol.31, No.4, 2013, 370–381.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1301-m4234

BLOCK-SYMMETRIC AND BLOCK-LOWER-TRIANGULAR PRECONDITIONERS FOR PDE-CONSTRAINED OPTIMIZATION PROBLEMS*

Guofeng Zhang and Zhong Zheng
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email: gf_zhang@lzu.edu.cn 463151643@qq.com

Abstract

Optimization problems with partial differential equations as constraints arise widely in many areas of science and engineering, in particular in problems of the design. The solution of such class of PDE-constrained optimization problems is usually a major computational task. Because of the complexion for directly seeking the solution of PDE-constrained optimization problem, we transform it into a system of linear equations of the saddle-point form by using the Galerkin finite-element discretization. For the discretized linear system, in this paper we construct a block-symmetric and a block-lower-triangular preconditioner, for solving the PDE-constrained optimization problem. Both preconditioners exploit the structure of the coefficient matrix. The explicit expressions for the eigenvalues and eigenvectors of the corresponding preconditioned matrices are derived. Numerical implementations show that these block preconditioners can lead to satisfactory experimental results for the preconditioned GMRES methods when the regularization parameter is suitably small.

Mathematics subject classification: 65F10, 65F50, 65F08, 65F22, 65F35, 65N22.
Key words: Saddle-point matrix, Preconditioning, PDE-constrained optimization, Eigenvalue and eigenvector, Regularization parameter.

1. Introduction

We consider the distributed control problem which consists of a cost functional (1.1) to be minimized subject to a partial differential equation problem posed on a domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$:

$$\min_{u,f} \frac{1}{2}\|u-u_*\|_2^2 + \beta\|f\|_2^2, \tag{1.1}$$

$$\text{subject to} \quad -\nabla^2 u = f \qquad \text{in } \Omega, \tag{1.2}$$

$$\text{with} \quad u_* = g \quad \text{on} \quad \partial\Omega_1 \quad \text{and} \quad \frac{\partial u_*}{\partial n} = g \quad \text{on} \quad \partial\Omega_2, \tag{1.3}$$

where $\partial\Omega$ is the boundary of Ω, $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, $\beta \in R^+$ is a regularization parameter, and the function u_* is a given function that represents the desired state. We want to find u which satisfies the PDE problem and is as close to u_* as possible in some norm sense (e.g., the L_2 norm). In order to achieve this aim, the right-hand side f of the PDE can be varied. The second term in the cost functional (1.1) is added because the problem would be generally ill-posed and then needs this Tikhonov regularization term. Such class of problems

* Received August 3, 2012 / Revised version received December 26, 2012 / Accepted January 31, 2013 / Published online July 9, 2013 /

were introduced by J.L. Lions in [15].

There are two approaches to obtain the solution of the PDE-constrained optimization problem (1.1)-(1.3). The first is optimize-then-discretize and the second is discretize-then-optimize. Following the discretize-then-optimize approach (see [1,16]), we transform (1.1)-(1.3) into a linear system of the saddle-point form. That is to say, firstly, by employing the Galerkin finite-element method to the weak formulation of (1.2) and (1.3), we obtain the finite-dimensional discrete analogue of the minimization problem as follows (see [1,11,13,15,16]):

$$\min_{u,f} \quad \frac{1}{2}u^T Mu - u^T b + \alpha + \beta f^T Mf, \tag{1.4}$$

$$\text{subject to} \quad Ku = Mf + d, \tag{1.5}$$

where $M \in \mathbb{R}^{n\times n}$ is the mass matrix, $K \in \mathbb{R}^{n\times n}$ is the stiffness matrix (the discrete Laplacian), $d \in \mathbb{R}^n$ represents the boundary data, $\alpha = ||u_*||_2^2$, and $b \in \mathbb{R}^n$ is the Galerkin projection of the discrete state u_*. Then by applying the Lagrangian multiplier method to this minimization problem (1.4)-(1.5) we find that f, u and λ are defined by the linear system

$$\mathcal{A}x \equiv \begin{pmatrix} 2\beta M & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{pmatrix} \begin{pmatrix} f \\ u \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ d \end{pmatrix} \equiv g, \tag{1.6}$$

where λ is a vector of Lagrange multiplier, see [1,16-17]. Evidently, if we let

$$A = \begin{pmatrix} 2\beta M & 0 \\ 0 & M \end{pmatrix}, \quad B = \begin{pmatrix} -M & K \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 & b^T \end{pmatrix}^T,$$

then the system of linear equations (1.6) can be transformed into the standard saddle-point system:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}. \tag{1.7}$$

Frequently, iterative methods are more attractive than direct methods for solving the saddle point problem (1.7), because the coefficient matrix of the saddle point problem (1.7) is large and sparse. Many efficient iterative methods have been studied in the literatures. For example, Uzawa-like methods ([8,10,12]), SOR-like methods ([7,14]), RPCG methods ([6, 9]), HSS-like methods ([2-5]) and so on. We refer to [2] for algebraic properties for saddle point problem (1.7). In this paper, we will focus on the systems that arise in the context of PDE-constrained optimization. Systems of the type given in (1.6) are typically very poorly conditioned and large sparse. Therefore preconditioning is usually necessary in practice in order to achieve rapid convergence of Krylov subspace methods.

In this paper, by exploiting the structure of the coefficient matrix, we construct a block-symmetric preconditioner and a block–lower-triangular preconditioner. The explicit expressions for the eigenvalues and eigenvectors of the corresponding preconditioned matrices are derived. Both theoretical analysis and numerical results show that the preconditioned GMRES(20) methods with these block preconditioners are effective and robust linear solvers for the saddle-point problems such as (1.6) from PDE-constrained optimization.

The organization of the paper is as follows. In Sections 2 and 3, we use the structure of the linear system (1.6) to give two block preconditioners. The explicit expressions for the eigenvalues of the two preconditioned matrices are derived. Numerical examples are given in Section 4 to show the effectiveness of these new preconditioners. Finally, we draw some conclusions in Section 5.