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A PARAMETER-UNIFORM TAILORED FINITE POINT METHOD FOR SINGULARLY PERTURBED LINEAR ODE SYSTEMS*

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Abstract

In scientific applications from plasma to chemical kinetics, a wide range of temporal scales can present in a system of differential equations. A major difficulty is encountered due to the stiffness of the system and it is required to develop fast numerical schemes that are able to access previously unattainable parameter regimes. In this work, we consider an initial-final value problem for a multi-scale singularly perturbed system of linear ordinary differential equations with discontinuous coefficients. We construct a tailored finite point method, which yields approximate solutions that converge in the maximum norm, uniformly with respect to the singular perturbation parameters, to the exact solution. A parameter-uniform error estimate in the maximum norm is also proved. The results of numerical experiments, that support the theoretical results, are reported.

Mathematics subject classification: 37M05, 65G99.
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1. Introduction

We consider the following initial-final value problem for a system of linear ordinary differential equations with discontinuous coefficients

$$\mathcal{E}\mathbf{u}'(t)+\mathcal{A}(t)\mathbf{u}(t)=\mathbf{f}(t), \quad \forall t\in(p_k,\, p_{k+1}), \quad k=0,\dots,K, \tag{1.1}$$
$$\mathbf{u}(p_k+0)-\mathbf{u}(p_k-0)=0, \quad k=1,\dots,K, \tag{1.2}$$
$$\mathcal{B}^{\varepsilon}\mathbf{u}(0)+\left(\mathbf{I}-\mathcal{B}^{\varepsilon}\right)\mathbf{u}(1)=\mathbf{d}, \tag{1.3}$$

where $\mathcal{E}$, $\mathcal{B}^{\varepsilon}$ are $n\times n$ matrices, $\mathbf{d}$ is a vector, $\mathcal{A}(t)$ is an $n\times n$ matrix function and $\mathbf{f}(t)$ is a vector function on the interval $[0,1]$ such that

$$\mathcal{A}(t)=\big(a_{i,j}(t)\big)_{n\times n}, \tag{1.4}$$
$$\mathbf{f}(t)=\left(f_1(t),f_2(t),\dots,f_n(t)\right)^T, \tag{1.5}$$

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and $\{p_k\}_0^{K+1}$ are some numbers which satisfy

$$0 = p_0 < p_1 < \cdots < p_K < p_{K+1} = 1.$$

The given functions $a_{i,j}(t), f_i(t)$ $(1 \le i, j \le n)$ may not be continuous on the whole interval $[0,1]$. Here, we consider the case when they are piecewise continuous. More precisely, we assume that the functions $a_{i,j}(t), f_i(t)$ $(1 \le i, j \le n)$ have K points of discontinuity of the first kind, $t = p_k, (0 < p_k < 1; k = 1, \dots, K)$, so that on each subinterval (p_k, p_{k+1}), $(k = 0, \dots, K; p_0 = 0, p_{K+1} = 1)$ the functions are smooth and satisfy the conditions

$$a_{i,i}(t) - \sum_{j=1, j\neq i}^{n} |a_{i,j}(t)| \ge \beta > 0, \quad \forall t \in [0,1]. \tag{1.6}$$

Furthermore, we assume that the matrices $\mathcal{E}^\varepsilon = diag(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $\mathcal{B}^\varepsilon = diag(b_1, b_2, \dots, b_n)$ are diagonal and satisfy the conditions

$$|\varepsilon_i| > 0, \ \ 1 \le i \le n, \tag{1.7}$$

$$b_i = \begin{cases} 1, & \varepsilon_i > 0, \\ 0, & \varepsilon_i < 0. \end{cases} \tag{1.8}$$

We also suppose that there exists at least one ε_j $(1 \le j \le n)$ such that

$$0 < |\varepsilon_j| \ll 1. \tag{1.9}$$

Problem (1.1)-(1.3) is then an initial-final value problem for a multi-scale singularly perturbed system of linear ordinary differential equations with discontinuous coefficients. The solution of problem (1.1)-(1.3) may contain initial, final and interior layers at any of the points p_k $(k = 1, \dots, K)$. The main goal in this paper is to develop a class of numerical methods, which yield approximate solutions that converge in the maximum norm, uniformly with respect to the singular perturbation parameters, to the exact solution of this problem.

When all of the parameters $(\varepsilon_j, \ j = 1, \dots, n)$ are positive, problem (1.1)-(1.3) reduces to the initial value singularly perturbed problem

$$\mathcal{L}\mathbf{u}(t) \equiv \mathcal{E}\mathbf{u}'(t) + \mathcal{A}(t)\mathbf{u}(t) = \mathbf{f}(t), \quad \forall t \in (p_k,\, p_{k+1}), \quad k = 0, \dots, K, \tag{1.10}$$

$$\mathbf{u}(p_k + 0) - \mathbf{u}(p_k - 0) = 0, \qquad k = 1, \dots, K, \tag{1.11}$$

$$\mathbf{u}(0) = \mathbf{d}, \tag{1.12}$$

which has been studied in [23].They proposed a Shishkin piecewise uniform mesh with a classical finite difference scheme to obtain numerical solutions of this problem; a parameter-uniform error estimate was also given.

To motivate the study of the more general initial-final value problem in the paper, it should be noted that a semi-discretization, with respect to variable x, of the following forward-backward parabolic problem

$$\mathrm{sign}(x)|x|^p \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \sigma(x)u(x) + q(x,t), \quad -1 < x < 2,\, 0 < t < T, \tag{1.13}$$

$$u\big|_{x=-1} = f(t), \qquad u\big|_{x=2} = g(t), \quad 0 < t < T, \tag{1.14}$$

$$u\big|_{t=0} = s(x), \ \ 0 < x < 2, \qquad u\big|_{t=1} = \gamma(x), \ \ -1 < x < 0, \tag{1.15}$$