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# FINITE VOLUME SUPERCONVERGENCE APPROXIMATION FOR ONE-DIMESIONAL SINGULARLY PERTURBED PROBLEMS*

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**Abstract**

We analyze finite volume schemes of arbitrary order $r$ for the one-dimensional singularly perturbed convection-diffusion problem on the Shishkin mesh. We show that the error under the energy norm decays as $(N^{-1}\ln(N+1))^r$, where $2N$ is the number of subintervals of the primal partition. Furthermore, at the nodal points, the error in function value approximation super-converges with order $(N^{-1}\ln(N+1))^{2r}$, while at the Gauss points, the derivative error super-converges with order $(N^{-1}\ln(N+1))^{r+1}$. All the above convergence and superconvergence properties are independent of the perturbation parameter $\epsilon$. Numerical results are presented to support our theoretical findings.

*Mathematics subject classification:* 65N30, 65N12, 65N06.
*Key words:* Finite Volume, High Order, Superconvergence, Convection-Diffsuion.

## 1. Introduction

We are interested in numerical solutions of singularly perturbed problems (SPP), whose approximation schemes are difficult to construct due to the effect of *boundary layers*. The subject has attracted much attention in scientific computing community (see, e.g., [2, 18, 19, 22, 24, 25, 29, 31, 32]). However, most theoretical studies in the literature have been focused on finite element methods (FEM) including discontinuous Galerkin (DG) methods.

On the other hand, the finite volume method (FVM) also has wide range of applications due to its local conservation of numerical fluxes (a property not shared by FEM), the capability of handling domains with complex geometries (a property shared by FEM), and other advantages, see, e.g., [3–6, 9, 12–14, 20, 21, 26, 30, 35]. Recently, FV schemes of arbitrary order have been constructed and analyzed for the two-point boundary value problem [7]. In this paper, we extend our study along this line to singularly perturbed problems. Note that traditional numerical methods on quasi-uniform meshes for SPP may be unstable and fail to give expected results.

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Therefore, we construct our FV schemes on the Shishkin type meshes ( [24]), which are well-known to be effective for the finite element approximation of SPP. Moreover, following [7], we use the Gauss points of the primal mesh to construct control volumes. Note that this idea of control volumes construction was used in some low-order FV schemes, see e.g. [17, 21, 27].

The special feature in the analysis for SPP is to establish $\epsilon$-independent error bounds. Therefore, the proof of the inf-sup condition is much more involved and special care must be taken. Similar to the finite element method, the FVM bilinear form for convection-diffusion problems is not uniformly continuous with respect to the singular perturbation parameter $\epsilon$ (see Section 3). To overcome this difficulty, we prove a *weak* continuity instead. With the inf-sup condition and weak continuity in hands, we prove that the approximation error under the energy norm has a near optimal order $(N^{-1}\ln(N+1))^r$.

We further investigate superconvergence properties of our finite volume schemes. Note that superconvergence properties of other numerical methods for SPP have been studied before, e.g., see [31] for finite element methods, [8, 33] for streamline diffusion finite element methods (SDFEM), and [28, 29] for DG methods. In this work, we establish a superconvergence rate of $(N^{-1}\ln(N+1))^{r+1}$ for our FVM under a discrete energy norm, similar to the result in [31] for the counterpart finite element method. As a direct consequence, a near optimal convergence rate in the $L^2$ norm is obtained. Finally, we prove nodal points superconvergence rate $(N^{-1}\ln(N+1))^{2r}$, which is similar to the one for SDFEM in [8]. We should point out that all aforementioned error bounds are independent of the singular perturbation parameter $\epsilon$. Moreover, our numerical data indicate that the logarithmic factors appeared in the estimates are not removable, and hance, our error bounds are sharp.

The outline of the rest of this paper is as follows. In Section 2, we present our FV schemes for the one-dimensional singularly perturbed convection-diffusion problem on the Shishkin mesh. In Section 3, we prove the inf-sup condition and a weak continuity and use them to establish the optimal convergence rate under the energy norm. In Section 4, we analyze superconvergence properties. Numerical results supporting our theoretical findings are provided in Section 5.

In the rest of this paper, "$A \lesssim B$" means that $A$ can be bounded by $B$ multiplied by a constant which is independent of $\epsilon$ and $N$. "$A \sim B$" stands for "$A \lesssim B$" and "$B \lesssim A$".

## 2. FV Schemes for Convection-Diffusion Problems

In this section, we introduce a family of finite volume schemes of arbitrary order to approximate the following convection-diffusion model problem.

$$-\epsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \ \forall x \in \Omega = (0,1), \tag{2.1a}$$
$$u(0) = u(1) = 0, \tag{2.1b}$$

where $0 < \epsilon \ll 1$ is a small positive parameter and

$$p(x) \geq p_0 > 0, \quad q(x) \geq q_0 > 0, \quad \forall x \in \bar{\Omega}.$$

There is no essential loss of generality to consider the following problem

$$-\epsilon a(x)u''(x) + u'(x) + b(x)u(x) = f(x), \ \forall x \in \Omega = (0,1), \tag{2.2a}$$
$$u(0) = u(1) = 0 \tag{2.2b}$$