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ESTIMATIONS OF THE CONSTANTS IN INVERSE INEQUALITIES FOR FINITE ELEMENT FUNCTIONS*

Shaochun Chen and Jikun Zhao
Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China
Email: shchchen@zzu.edu.cn zjk021466@163.com

Abstract

In this paper, we estimate the constants in the inverse inequalities for the finite element functions. Furthermore, we obtain the least upper bounds of the constants in inverse inequalities for the low-order finite element functions. Such explicit estimates of the constants can be used as computable error bounds for the finite element method.

Mathematics subject classification: 65N30.
Key words: Inverse inequality, Finite element function, Bound of the constant.

1. Introduction

As a very effective numerical method of partial differential equations, the finite element method (FEM) is widely applied to the engineering and scientific computation. In the process of analysis and solving by the finite element method, the inverse inequalities are frequently used to bound the high-order (semi-)norms in terms of the low-order ones for the finite element functions, cf. [1, 3]. However, it is well known to prove the inverse inequalities by functional analysis [3], which can not explicitly give the constants on the right and brings troubles into practical error analysis and numerical computation such as a posteriori error estimation and adaptive refinement algorithms. Therefore, for both error analysis and numerical computation, it is very significant to estimate the constants in the inverse inequalities.

We consider the following inverse inequality

$$|v|_{1,\Omega} \leq Ch^{-1}\|v\|_{\Omega}, \tag{1.1}$$

where h is the diameter of the domain Ω and v is a finite element function. The other kinds of inverse inequalities are considered in [4, 5, 7, 10].

For the 1-D and 2-D cases, the constant C in (1.1) is given for the linear finite element in [2]. For any dimension n and order k, The estimation on the constant C in (1.1) is translated into a conditional extremum problem in [9]. However, it needs to solve a system of nonlinear equations with the help of the software Matlab, which is not suitable for theoretic analysis.

In this paper, we explicitly give the inverse inequalities for the finite element functions by different methods. In section 2, due to the recursion relation and orthogonality of Legendre polynomials [8], the constant C in (1.1) is estimated for the 1-D case, which may be extended to general rectangular domains. Especially, we obtained the optimal constants for $k = 1, 2$. In section 3, the constant C in (1.1) is estimated for the reference triangular and tetrahedron finite elements, respectively, which can be ordinarily extended to general bounded domains.

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Furthermore, we obtain the least upper bound of the constant for the linear triangular and tetrahedron finite elements, respectively. In addition, we get an explicit relation between the inverse inequality (1.1) and the geometric characters of the general triangle T. Finally, from the inverse inequality (1.1) we explicitly obtain general inverse inequalities as follows

$$|v|_{m,\Omega} \le Ch^{l-m}|v|_{l,\Omega}, \tag{1.2}$$

where $|v|_{m,\Omega} = \big(\sum\limits_{|\alpha|=m} \frac{|\alpha|!}{\alpha!}\|D^\alpha v\|^2_{\Omega}\big)^{\frac{1}{2}}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n$ and $m \ge 1$.

2. Inverse Inequalities for the 1-D Case

In this paper, we denote a polynomial space of order $\le k$ on Ω by $P_k(\Omega)$. Let L_k be the k-th Legendre polynomial, that is,

$$L_i(x) = \frac{1}{2^i i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}(x^2-1)^i, \quad i = 0, 1, \ldots, k, \tag{2.1}$$

with the following orthogonality

$$(L_i, L_j) \triangleq \int_{-1}^{1} L_i(x)L_j(x)\mathrm{d}x = \begin{cases} 0, & i \ne j, \\ \frac{2}{2i+1}, & i = j. \end{cases} \tag{2.2}$$

For $1 \le i \le k$, according to (2.1) we have the following recursion formula

$$\begin{aligned} L_i'(x) &= \frac{1}{2^i i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}\Big(2ix(x^2-1)^{i-1}\Big) \\ &= \frac{1}{2^{(i-1)}(i-1)!}\frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}\Big((x^2-1)^{i-1} + 2(i-1)x^2(x^2-1)^{i-2}\Big) \\ &= \frac{1}{2^{(i-1)}(i-1)!}\frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}\Big((2i-1)(x^2-1)^{i-1} + 2(i-1)(x^2-1)^{i-2}\Big) \\ &= (2i-1)L_{i-1}(x) + L_{i-2}'(x). \end{aligned}$$

Then there holds

$$L_i'(x) = \sum_{j=0}^{i-1} d_{ij}L_j(x) = \sum_{j=0}^{k-1} d_{ij}L_j(x), \tag{2.3}$$

where

$$d_{ij} = \begin{cases} 2j+1, & \text{if } i-j \text{ is odd and positive}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

From (2.3) and (2.4), we have the following orthogonality relation

$$(L_{2i}', L_{2j-1}') = 0. \tag{2.5}$$

For any $p(x) \in P_k(-1,1)$, there exist $k+1$ real numbers $c_0, c_1, \ldots, c_k$ such that

$$p(x) = \sum_{i=0}^{k} c_i L_i(x), \tag{2.6}$$

$$\|p\|^2_{L^2(-1,1)} = \Big(\sum_{i=0}^{k} c_iL_i, \sum_{i=0}^{k} c_iL_i\Big) = \sum_{i=0}^{k} c_i^2(L_i, L_i) = \sum_{i=0}^{k} \frac{2c_i^2}{2i+1}. \tag{2.7}$$