Journal of Computational Mathematics
Vol.31, No.5, 2013, 488–508.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1304-m4280

FINITE VOLUME SUPERCONVERGENCE APPROXIMATION FOR ONE-DIMESIONAL SINGULARLY PERTURBED PROBLEMS*

Waixiang Cao
School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou 510275, China
Email: ziye101@163.com
Zhimin Zhang
Beijing Computational Science Research Center, Beijing 100084, China
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
Email: ag7761@wayne.edu
Qingsong Zou
School of Mathematics and Computational Science and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen University, Guangzhou 510275, China
Email:mcszqs@mail.sysu.edu.cn

Abstract

We analyze finite volume schemes of arbitrary order r for the one-dimensional singularly perturbed convection-diffusion problem on the Shishkin mesh. We show that the error under the energy norm decays as $(N^{-1}\ln(N+1))^r$, where $2N$ is the number of subintervals of the primal partition. Furthermore, at the nodal points, the error in function value approximation super-converges with order $(N^{-1}\ln(N+1))^{2r}$, while at the Gauss points, the derivative error super-converges with order $(N^{-1}\ln(N+1))^{r+1}$. All the above convergence and superconvergence properties are independent of the perturbation parameter ϵ. Numerical results are presented to support our theoretical findings.

Mathematics subject classification: 65N30, 65N12, 65N06.
Key words: Finite Volume, High Order, Superconvergence, Convection-Diffsuion.

1. Introduction

We are interested in numerical solutions of singularly perturbed problems (SPP), whose approximation schemes are difficult to construct due to the effect of *boundary layers*. The subject has attracted much attention in scientific computing community (see, e.g., [2, 18, 19, 22, 24, 25, 29, 31, 32]). However, most theoretical studies in the literature have been focused on finite element methods (FEM) including discontinuous Galerkin (DG) methods.

On the other hand, the finite volume method (FVM) also has wide range of applications due to its local conservation of numerical fluxes (a property not shared by FEM), the capability of handling domains with complex geometries (a property shared by FEM), and other advantages, see, e.g., [3–6, 9, 12–14, 20, 21, 26, 30, 35]. Recently, FV schemes of arbitrary order have been constructed and analyzed for the two-point boundary value problem [7]. In this paper, we extend our study along this line to singularly perturbed problems. Note that traditional numerical methods on quasi-uniform meshes for SPP may be unstable and fail to give expected results.

* Received October 1, 2012 / Revised version received April 1, 2013 / Accepted April 16, 2013 / Published online August 27, 2013 /

Therefore, we construct our FV schemes on the Shishkin type meshes ([24]), which are well-known to be effective for the finite element approximation of SPP. Moreover, following [7], we use the Gauss points of the primal mesh to construct control volumes. Note that this idea of control volumes construction was used in some low-order FV schemes, see e.g. [17, 21, 27].

The special feature in the analysis for SPP is to establish ϵ-independent error bounds. Therefore, the proof of the inf-sup condition is much more involved and special care must be taken. Similar to the finite element method, the FVM bilinear form for convection-diffusion problems is not uniformly continuous with respect to the singular perturbation parameter ϵ (see Section 3). To overcome this difficulty, we prove a *weak* continuity instead. With the inf-sup condition and weak continuity in hands, we prove that the approximation error under the energy norm has a near optimal order $(N^{-1}\ln(N+1))^r$.

We further investigate superconvergence properties of our finite volume schemes. Note that superconvergence properties of other numerical methods for SPP have been studied before, e.g., see [31] for finite element methods, [8, 33] for streamline diffusion finite element methods (SDFEM), and [28, 29] for DG methods. In this work, we establish a superconvergence rate of $(N^{-1}\ln(N+1))^{r+1}$ for our FVM under a discrete energy norm, similar to the result in [31] for the counterpart finite element method. As a direct consequence, a near optimal convergence rate in the L^2 norm is obtained. Finally, we prove nodal points superconvergence rate $(N^{-1}\ln(N+1))^{2r}$, which is similar to the one for SDFEM in [8]. We should point out that all aforementioned error bounds are independent of the singular perturbation parameter ϵ. Moreover, our numerical data indicate that the logarithmic factors appeared in the estimates are not removable, and hance, our error bounds are sharp.

The outline of the rest of this paper is as follows. In Section 2, we present our FV schemes for the one-dimensional singularly perturbed convection-diffusion problem on the Shishkin mesh. In Section 3, we prove the inf-sup condition and a weak continuity and use them to establish the optimal convergence rate under the energy norm. In Section 4, we analyze superconvergence properties. Numerical results supporting our theoretical findings are provided in Section 5.

In the rest of this paper, "$A \lesssim B$" means that A can be bounded by B multiplied by a constant which is independent of ϵ and N. "$A \sim B$" stands for "$A \lesssim B$" and "$B \lesssim A$".

2. FV Schemes for Convection-Diffusion Problems

In this section, we introduce a family of finite volume schemes of arbitrary order to approximate the following convection-diffusion model problem.

$$-\epsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \ \forall x \in \Omega = (0,1), \tag{2.1a}$$

$$u(0) = u(1) = 0, \tag{2.1b}$$

where $0 < \epsilon \ll 1$ is a small positive parameter and

$$p(x) \geq p_0 > 0, \quad q(x) \geq q_0 > 0, \quad \forall x \in \bar{\Omega}.$$

There is no essential loss of generality to consider the following problem

$$-\epsilon a(x)u''(x) + u'(x) + b(x)u(x) = f(x), \ \forall x \in \Omega = (0,1), \tag{2.2a}$$

$$u(0) = u(1) = 0 \tag{2.2b}$$

with

$$a(x) \geq a_0 > 0, \quad b(x) \geq b_0 > 0, \quad \forall x \in \bar{\Omega}.$$

In fact, the above two equations are equivalent, since we can obtain (2.2) from (2.1) by multiplying (2.1) with $\frac{1}{p(x)}$.

By the regularity analysis ([18], Chapter 8), the exact solution u can be decomposed into

$$u = \bar{u} + u_\epsilon,$$

where the regular part $\bar{u}$ and the singular part u_ϵ satisfy:

$$\|\bar{u}^{(k)}\|_{L^\infty} \lesssim 1, \quad |u_\epsilon^{(k)}(x)| \lesssim \epsilon^{-k} e^{-\beta(1-x)/\epsilon}, \quad \forall x \in (0,1), \ k \geq 0. \tag{2.3}$$

Note that the exact solution u exhibits a boundary layer at $x = 1$.

We present our method under the framework of the Petrov-Galerkin method. We begin with the construction of the primary partition $\mathcal{P}$ and its corresponding trial space. Here a Shishkin type mesh is used as our primary partition $\mathcal{P}$ by introducing

$$\lambda = \min\Big(\frac{1}{2}, \frac{\epsilon}{\beta}(r+2)\ln(N+1)\Big),$$

and then dividing the intervals $(0, 1-\lambda)$ and $(1-\lambda, 1)$ into N equal-size subintervals. Hence, the element length in $(1-\lambda, 1)$ is $h_i = \underline{h} = \lambda/N$, whereas in $(0, 1-\lambda)$ is $h_i = \overline{h} = (1-\lambda)/N$. In this article, we shall only consider the case when

$$\frac{\epsilon}{\beta}(r+2)\ln(N+1) \leq \frac{1}{2},$$

for otherwise, r and N would be large enough to catch the boundary layer or the problem is regular. In either case, the traditional analysis would apply.

Let $0 = x_0 < x_1 < \ldots < x_{2N} = 1$ be $2N+1$ distinct points on the interval $\bar{\Omega}$. For all positive integer k, let $\mathbb{Z}_k = \{1, 2, \ldots, k\}$. Then

$$\mathcal{P} = \Big\{\tau_i | \tau_i = (x_{i-1}, x_i), i \in \mathbb{Z}_{2N}\Big\},$$

constitutes a partition of $\bar{\Omega}$.

The corresponding trial space is chosen as the Lagrange finite element of rth order, $r \geq 1$, defined by

$$U_{\mathcal{P}}^r = \Big\{v \in C(\Omega) : v|_\tau \in \mathbb{P}_r, \forall \tau \in \mathcal{P}, v|_{\partial\Omega} = 0\Big\},$$

where $\mathbb{P}_r$ is the set of all polynomials of degree no more than r. Obviously, $dim U_{\mathcal{P}}^r = 2Nr - 1$.

We next present the dual partition $\mathcal{P}'$ and its corresponding test space. Let $G_1, \ldots, G_r$ be r Gauss points, i.e., zeros of the Legendre polynomial of rth degree, on the interval $[-1, 1]$. The Gauss points on each interval $\tau_i, i \in \mathbb{Z}_{2N}$ are defined as the affine transformations of G_j to τ_i, that is :

$$g_{i,j} = \frac{1}{2}\Big(x_i + x_{i-1} + h_i G_j\Big), \quad j \in \mathbb{Z}_r.$$

With these Gauss points, we construct the dual partition

$$\mathcal{P}' = \{\tau'_{1,0}, \tau'_{2N,r}\} \cup \{\tau'_{i,j} : \tau'_{i,j} = [g_{i,j}, g_{i,j+1}], \ (i,j) \in \mathbb{Z}_{2N} \times \mathbb{Z}_{r_i}\},$$

where

$$\tau'_{1,0} = [0, g_{1,1}], \quad \tau'_{2N,r} = [g_{2N,r}, 1],$$

and

$$r_i = \begin{cases} r & \text{if} \quad i \in \mathbb{Z}_{2N-1}, \\ r-1 & \text{if} \quad i = 2N, \end{cases} \qquad g_{i,r+1} = g_{i+1,1}, \quad \forall i \in \mathbb{Z}_{2N-1}.$$

The test space $V_{\mathcal{P}'}$ consists of the piecewise constant functions with respect to the partition $\mathcal{P}'$, which vanish on the intervals $\tau'_{1,0} \cup \tau'_{2N,r}$. In other words,

$$V_{\mathcal{P}'} = \text{Span}\left\{\psi_{i,j} : (i,j) \in \mathbb{Z}_{2N} \times \mathbb{Z}_{r_i}\right\},$$

where $\psi_{i,j} = \chi_{[g_{i,j}, g_{i,j+1}]}$ is the characteristic function on the interval $\tau'_{i,j}$. Such a construction guarantees that $\dim V_{\mathcal{P}'} = 2Nr - 1 = \dim U^r_{\mathcal{P}}$.

We are ready to present our finite volume scheme. Integrating (2.2) on each control volume $[g_{i,j}, g_{i,j+1}], (i,j) \in \mathbb{Z}_{2N} \times \mathbb{Z}_{r_i}$ yields

$$\int_{g_{i,j}}^{g_{i,j+1}} -\epsilon a(x)u''(x) + u'(x) + b(x)u(x)dx = \int_{g_{i,j}}^{g_{i,j+1}} f(x)dx. \tag{2.4}$$

Let $w_{\mathcal{P}'} \in V_{\mathcal{P}'}$, $w_{\mathcal{P}'}$ can be represented as

$$w_{\mathcal{P}'} = \sum_{i=1}^{2N} \sum_{j=1}^{r_i} w_{i,j} \psi_{i,j},$$

where $w'_{i,j}s$ are constants. Multiplying (2.4) with $w_{i,j}$ and then summing up for all i, j, we obtain

$$\sum_{i=1}^{2N} \sum_{j=1}^{r_i} w_{i,j} \left(\int_{g_{i,j}}^{g_{i,j+1}} -\epsilon a(x)u''(x) + u'(x) + b(x)u(x)dx \right) = \int_0^1 f(x) w_{\mathcal{P}'}(x)dx,$$

or equivalently,

$$\begin{aligned} &\sum_{i=1}^{2N} \sum_{j=1}^{r} \left(\epsilon [w_{i,j}] a(g_{i,j}) u'(g_{i,j}) + w_{i,j} \int_{g_{i,j}}^{g_{i,j+1}} \big((\epsilon a'(x) + 1) u'(x) + b(x)u(x) \big) dx \right) \\ &= \int_0^1 f(x) w_{\mathcal{P}'}(x)dx, \end{aligned}$$

where $[w_{i,j}] = w_{i,j} - w_{i,j-1}$ is the jump of w at the point $g_{i,j}, (i,j) \in \mathbb{Z}_{2N} \times \mathbb{Z}_r$ with $w_{1,0} = 0, w_{2N,r} = 0$ and $w_{i,0} = w_{i-1,r}, 2 \leq i \leq 2N$.

We define the FVM bilinear form for all $v \in H^1_0(\Omega), w_{\mathcal{P}'} \in V_{\mathcal{P}'}$ by

$$\begin{aligned} a_{\mathcal{P}}(v, w_{\mathcal{P}'}) = &\sum_{i=1}^{2N} \sum_{j=1}^{r} \epsilon [w_{i,j}] a(g_{i,j}) v'(g_{i,j}) \\ &+ \sum_{i=1}^{2N} \sum_{j=1}^{r_i} w_{i,j} \left(\int_{g_{i,j}}^{g_{i,j+1}} \big((\epsilon a'(x) + 1) v'(x) + b(x) v(x) \big) dx \right). \end{aligned} \tag{2.5}$$

The finite volume method for solving the equation (2.2) reads as : Find $u_{\mathcal{P}} \in U^r_{\mathcal{P}}$ such that

$$a_{\mathcal{P}}(u_{\mathcal{P}}, w_{\mathcal{P}'}) = (f, w_{\mathcal{P}'}), \quad \forall w_{\mathcal{P}'} \in V_{\mathcal{P}'}. \tag{2.6}$$

3. Convergence

This section is devoted to the error estimate under the energy norm. An error bound of $(N^{-1}\ln(N+1))^r$ under the energy norm will be established. Our analysis is under the framework of Petrov-Galerkin method, which requires the establishment of the inf-sup condition and continuity of the bilinear form (2.5).

3.1. Inf-sup condition

We use the natural energy norm

$$\|v\|_\epsilon^2 = |v|_\epsilon^2 + (v,v), \quad |v|_\epsilon^2 = \epsilon(v', v'), \tag{3.1}$$

for all $v \in H_0^1(\Omega)$, and a discrete energy norm following [31]

$$\|v\|_{\epsilon,G}^2 = |v|_{\epsilon,G}^2 + (v,v), \quad |v|_{\epsilon,G}^2 = \epsilon \sum_{i=1}^{2N}\sum_{j=1}^{r} A_{i,j} v'(g_{i,j})^2, \tag{3.2}$$

where $A_{i,j}$s are weights for the r-point Gaussian quadrature on the interval τ_i. Since the r-point Gaussian quadrature is exact for polynomials of degree $2r-1$, then

$$\|v\|_\epsilon = \|v\|_{\epsilon,G}, \quad \forall v \in U_{\mathcal{P}}^r.$$

For all $w_{\mathcal{P}'} = \sum_{i=1}^{2N}\sum_{j=1}^{r_i} w_{ij}\psi_{i,j} \in V_{\mathcal{P}'}$, we define an ϵ-dependent semi-norm by

$$|w_{\mathcal{P}'}|_{\mathcal{P}',\epsilon}^2 = \epsilon \sum_{i=1}^{2N}\sum_{j=1}^{r} h_i^{-1}[w_{i,j}]^2, \tag{3.3}$$

and a norm by

$$\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}^2 = |w_{\mathcal{P}'}|_{\mathcal{P}',\epsilon}^2 + \sum_{i=1}^{2N}\sum_{j=1}^{r_i} h_i w_{i,j}^2. \tag{3.4}$$

To discuss the relationship of the norms between the trial and test spaces, we recall the linear mapping from the trial space to the test space introduced in [7]. Let $\Pi_{\mathcal{P}} : U_{\mathcal{P}}^r \to V_{\mathcal{P}'}$ be the mapping defined for all $w_{\mathcal{P}} \in U_{\mathcal{P}}^r$ by

$$\Pi_{\mathcal{P}} w_{\mathcal{P}} := w_{\mathcal{P}'} = \sum_{i=1}^{2N}\sum_{j=1}^{r_i} w_{i,j}\psi_{i,j},$$

where the coefficients $w_{i,j}$ are determined by the constraints

$$[w_{i,j}] = A_{i,j} w'_{\mathcal{P}}(g_{i,j}), \quad (i,j) \in \mathbb{Z}_{2N} \times \mathbb{Z}_{r_i}.$$

It is shown in [7] that

$$[w_{2N,r}] = A_{2N,r} w'_{\mathcal{P}}(g_{2N,r}).$$

Consequently,

$$|w_{\mathcal{P}}|_{1,\tau_i}^2 \sim \sum_{j=1}^{r} h_i^{-1}[w_{i,j}]^2, \quad \forall w_{\mathcal{P}} \in U_{\mathcal{P}}^r, \quad \forall i \in \mathbb{Z}_{2N}. \tag{3.5}$$

We next show that a similar equivalence holds for the ϵ-dependent energy norm.

Lemma 3.1. *For all $w_{\mathcal{P}} \in U^r_{\mathcal{P}}$, there holds*

$$\|w_{\mathcal{P}}\|_{\epsilon} \sim \|\Pi_{\mathcal{P}} w_{\mathcal{P}}\|_{\mathcal{P}',\epsilon}.$$

Proof. By (3.5) and the definitions of $\|\cdot\|_{\mathcal{P}',\epsilon}$ and $\|\cdot\|_{\epsilon}$, we only need to prove

$$\|w_{\mathcal{P}}\|_0^2 \sim \sum_{i=1}^{2N} \sum_{j=1}^{r_i} h_i w_{i,j}^2. \tag{3.6}$$

Noticing that for all $w_{\mathcal{P}} \in U^r_{\mathcal{P}}$

$$w_{\mathcal{P}}(x) = \int_{x_{i-1}}^{x} w'_{\mathcal{P}}(t)dt + w_{\mathcal{P}}(x_{i-1}), \quad \forall x \in \tau_i, \quad i \in \mathbb{Z}_{2N},$$

then

$$\|w_{\mathcal{P}}\|_{0,\tau_i}^2 \lesssim h_i w_{\mathcal{P}}^2(x_{i-1}) + h_i^2 |w_{\mathcal{P}}|_{1,\tau_i}^2.$$

Similarly, we have

$$h_i w_{\mathcal{P}}^2(x_{i-1}) \lesssim \|w_{\mathcal{P}}\|_{0,\tau_i}^2 + h_i^2 |w_{\mathcal{P}}|_{1,\tau_i}^2.$$

By the inverse inequality,

$$h_i w_{\mathcal{P}}^2(x_{i-1}) + h_i^2 |w_{\mathcal{P}}|_{1,\tau_i}^2 \lesssim \|w_{\mathcal{P}}\|_{0,\tau_i}^2 + h_i^2 |w_{\mathcal{P}}|_{1,\tau_i}^2 \lesssim \|w_{\mathcal{P}}\|_{0,\tau_i}^2.$$

Consequently,

$$\|w_{\mathcal{P}}\|_{0,\tau_i}^2 \sim h_i w_{\mathcal{P}}^2(x_{i-1}) + h_i^2 |w_{\mathcal{P}}|_{1,\tau_i}^2, \quad \forall i \in \mathbb{Z}_{2N}. \tag{3.7}$$

Note that

$$w_{\mathcal{P}}(x_{i-1}) = \int_0^{x_{i-1}} w'_{\mathcal{P}}(x)dx = \sum_{k=1}^{i-1} \sum_{j=1}^{r} [w_{k,j}] = w_{i-1,r} = w_{i,0}.$$

Then by (3.5) and (3.7), we have

$$\|w_{\mathcal{P}}\|_{0,\tau_i}^2 \backsim h_i w_{i,0}^2 + h_i^2 \sum_{j=1}^{r} h_i^{-1} [w_{i,j}]^2 \lesssim h_i \sum_{j=0}^{r} w_{i,j}^2.$$

On the other hand,

$$\begin{aligned} h_i \sum_{j=0}^{r} w_{i,j}^2 &\lesssim h_i w_{i,0}^2 + h_i \sum_{j=1}^{r} \sum_{k=1}^{j} [w_{i,k}]^2 \\ &\lesssim h_i w_{i,0}^2 + h_i^2 |w_{\mathcal{P}}|_{1,\tau_i}^2 \lesssim |w_{\mathcal{P}}|_{0,\tau_i}^2. \end{aligned}$$

In summary, we have

$$\|w_{\mathcal{P}}\|_{0,\tau_i}^2 \sim h_i \sum_{j=0}^{r} w_{i,j}^2, \quad \forall i \in \mathbb{Z}_{2N}.$$

Summing up the above equivalence for all i gives (3.6). The conclusion follows immediately. □

With all these preparations, we are now ready to present the inf-sup property of $a_{\mathcal{P}}(\cdot,\cdot)$.

Theorem 3.1. *Assume that the mesh size $\overline{h}$ is sufficiently small, then*

$$\inf_{v_{\mathcal{P}}\in U^r_{\mathcal{P}}}\sup_{w_{\mathcal{P}'}\in V_{\mathcal{P}'}}\frac{a_{\mathcal{P}}(v_{\mathcal{P}},w_{\mathcal{P}'})}{\|v_{\mathcal{P}}\|_\epsilon\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}}\ge\beta_0, \tag{3.8}$$

where $\beta_0>0$ is a constant independent of ϵ and N.

Proof. Recall the bilinear form (2.5), for all $v_{\mathcal{P}}\in U^r_{\mathcal{P}}$, we have

$$a_{\mathcal{P}}(v_{\mathcal{P}},\Pi_{\mathcal{P}}v_{\mathcal{P}})=J_1+J_2+J_3,$$

with

$$\begin{aligned}
J_1&=\epsilon\sum_{i=1}^{2N}\sum_{j=1}^{r}[v_{i,j}]a(g_{i,j})v'_{\mathcal{P}}(g_{i,j}),\\
J_2&=\sum_{i=1}^{2N}\sum_{j=1}^{r_i}v_{i,j}\int_{g_{i,j}}^{g_{i,j+1}}(\epsilon a'(x)+1)v'_{\mathcal{P}}(x)dx,\\
J_3&=\sum_{i=1}^{2N}\sum_{j=1}^{r_i}v_{i,j}\int_{g_{i,j}}^{g_{i,j+1}}b(x)v_{\mathcal{P}}(x)dx.
\end{aligned}$$

Obviously,

$$J_1\ge a_0\epsilon\sum_{i=1}^{2N}\sum_{j=1}^{r}A_{i,j}(v'_{\mathcal{P}}(g_{i,j}))^2=a_0|v_{\mathcal{P}}|^2_\epsilon.$$

We now estimate J_2. By Young's inequality and (3.6), we have

$$\begin{aligned}
&\sum_{i=1}^{2N}\sum_{j=1}^{r_i}v_{i,j}\int_{g_{i,j}}^{g_{i,j+1}}\epsilon a'(x)v'_{\mathcal{P}}(x)dx\\
&\le\frac{a_0}{2}|v_{\mathcal{P}}|^2_\epsilon+c\epsilon\sum_{i=1}^{2N}\sum_{j=1}^{r_i}h_iv^2_{i,j}\le\frac{a_0}{2}|v_{\mathcal{P}}|^2_\epsilon+c_1\epsilon\|v_{\mathcal{P}}\|^2_0,
\end{aligned}$$

where c,c_1 are constants independent of ϵ and N. Note that

$$\sum_{i=1}^{2N}\sum_{j=1}^{r_i}v_{i,j}\int_{g_{i,j}}^{g_{i,j+1}}v'_{\mathcal{P}}(x)dx=-\sum_{i=1}^{2N}\sum_{j=1}^{r}[v_{i,j}]v_{\mathcal{P}}(g_{i,j})=\int_0^1v'_{\mathcal{P}}v_{\mathcal{P}}=0.$$

Then we have

$$J_2\ge-\frac{a_0}{2}|v_{\mathcal{P}}|^2_\epsilon-c_1\epsilon\|v_{\mathcal{P}}\|^2_0.$$

As for J_3, we let $V(x)=\int_0^x b(s)v_{\mathcal{P}}(s)ds, x\in\Omega$ and

$$E_i=\int_{x_{i-1}}^{x_i}v'_{\mathcal{P}}(x)V(x)dx-\sum_{j=1}^{r}A_{i,j}v'_{\mathcal{P}}(g_{i,j})V(g_{i,j}),$$

be the error of Gauss quadrature on the interval τ_i, $i\in\mathbb{Z}_{2N}$. Then

$$\begin{aligned}
J_3&=-\sum_{i=1}^{2N}\sum_{j=1}^{r}[v_{i,j}]V(g_{i,j})=-\int_0^1v'_{\mathcal{P}}(x)V(x)dx+\sum_{i=1}^{2N}E_i\\
&=\int_0^1b(x)v^2_{\mathcal{P}}(x)dx+\sum_{i=1}^{2N}E_i.
\end{aligned}$$

It follows from [10] (p.98, (2.7.12)) that there exists a $\xi_i \in \tau_i, i \in \mathbb{Z}_{2N}$ such that

$$\begin{aligned}E_i &= \frac{h_i^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}(v'_{\mathcal{P}}V)^{(2r)}(\xi_i)\\ &= c_r h_i^{2r+1}\left(b(\xi_i)(v_{\mathcal{P}}^{(r)}(\xi_i))^2 + \sum_{k=0}^{r-2}\binom{2r}{k}(v'_{\mathcal{P}})^{(k)}(\xi_i)V^{(2r-k)}(\xi_i)\right).\end{aligned}$$

By the inverse inequality, for all $k \in \mathbb{Z}_{r-2}$

$$\begin{aligned}&\left|(v'_{\mathcal{P}})^{(k)}(\xi_i)V^{(2r-k)}(\xi_i)\right|\\ &\le c_{r,b}\sum_{j+k\le 2r-1}|v_{\mathcal{P}}|_{j,\infty,\tau_i}|v_{\mathcal{P}}|_{k,\infty,\tau_i} \le c'_{r,b}\sum_{j+k\le 2r-1} h_i^{-(j+k+1)}\|v_{\mathcal{P}}\|^2_{0,\tau_i},\end{aligned}$$

where $c_{r,b}$ and $c'_{r,b}$ are constants dependent on r, b. Then

$$E_i \ge -ch_i\|v_{\mathcal{P}}\|^2_{0,\tau_i}, \quad \forall i \in \mathbb{Z}_{2N}.$$

Therefore, when $\overline{h}$ is sufficiently small, we have

$$J_3 \;=\; \int_0^1 b(x)v_{\mathcal{P}}^2(x)dx + \sum_{i=1}^{2N} E_i \ge \frac{b_0}{2}\|v_{\mathcal{P}}\|_0^2.$$

Consequently,

$$a_{\mathcal{P}}(v_{\mathcal{P}}, \Pi_{\mathcal{P}}v_{\mathcal{P}}) = J_1 + J_2 + J_3 \ge \frac{\alpha_0}{2}\|v_{\mathcal{P}}\|^2_{\epsilon}.$$

By Lemma 3.1, there holds for any $v_{\mathcal{P}} \in U^r_{\mathcal{P}}$,

$$\sup_{w_{\mathcal{P}'}\in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(v_{\mathcal{P}}, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}} \ge \frac{a_{\mathcal{P}}(v_{\mathcal{P}}, \Pi_{\mathcal{P}}v_{\mathcal{P}})}{\|\Pi_{\mathcal{P}}v_{\mathcal{P}}\|_{\mathcal{P}',\epsilon}} \ge \beta_0\|v_{\mathcal{P}}\|_{\epsilon},$$

with β_0 independent of ϵ and N. The inf-sup property (3.8) follows. □

3.2. On the continuity

Under the framework of Petrov-Galerkin method, the convectional continuity of $a_{\mathcal{P}}(\cdot,\cdot)$ means that for all $v \in U^r_{\mathcal{P}}, w_{\mathcal{P}'} \in V_{\mathcal{P}'}$,

$$a_{\mathcal{P}}(v, w_{\mathcal{P}'}) \lesssim \|v\|_{\epsilon}\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}, \tag{3.9}$$

where the hidden constant should be independent of the small parameter ϵ. However, due to the existence of the term u' in (2.2), for convection-diffusion problems, (3.9) may not hold uniformly with respect to ϵ. Therefore in the following, we show the continuity property (3.12) which is slightly weaker than (3.9) but is sufficient for the establishment of our optimal error estimate.

We begin with a special interpolation. Let $l_{i,j}, (i,j) \in \mathbb{Z}_{2N}\times\mathbb{Z}_{r-1}$ be derivative zeros of the Legendre polynomial of degree r on the interval $\tau_i, i \in \mathbb{Z}_{2N}$, then $l_{i,j}$, together with $l_{i,0} = x_{i-1}$ and $l_{i,r} = x_i$, are called the Lobatto points of degree $r+1$ on the interval τ_i. Let u_I be a polynomial of degree r that interpolates u at those $r+1$ Lobatto points. Following the basic idea in [31], we choose the special interpolation of solution u as $I_\epsilon u = \bar{u}_I + u_{\epsilon,I_\epsilon}$, where

$$u_{\epsilon,I_\epsilon} = \begin{cases} u_{\epsilon,I}, & 1-\lambda \le x \le 1, \\ l_\lambda, & 1-\lambda-\bar{h} \le x \le 1-\lambda, \\ 0, & 0 \le x \le 1-\lambda-\bar{h}, \end{cases}$$

with

$$l_\lambda(x) = \begin{cases} u_\epsilon(1-\lambda)\frac{x-1+\lambda+\bar{h}}{\bar{h}}, & 1-\lambda-\bar{h} \le x \le 1-\lambda, \\ 0, & x \in (0, 1-\lambda-\bar{h}) \cup (1-\lambda, 1). \end{cases}$$

A direct calculation shows that

$$\|l_\lambda\|_0^2 = \bar{h}u_\epsilon(1-\lambda)^2/3 \lesssim \frac{1}{N^{2r+5}},$$
$$\|l_\lambda\|_1^2 = u_\epsilon(1-\lambda)^2/\bar{h} \lesssim \frac{1}{N^{2r+3}}.$$

We next present some approximate properties of $I_\epsilon u$. We omit the proof of Lemma 3.2 since the arguments are similar to those in [31]. The only difference is : here $\lambda = \frac{\epsilon}{\beta}(r+2)\ln(N+1)$ instead of $\lambda = \frac{\epsilon}{\beta}(r+1.5)\ln(N+1)$ in [31].

Lemma 3.2. *Let u_ϵ satisfy the regularity* (2.3). *Then*

$$\|u_\epsilon - u_{\epsilon,I}\|_{0,(1-\lambda,1)} \lesssim \sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^{r+1}, \quad \|u_\epsilon\|_{0,(0,1-\lambda)} \lesssim \frac{\sqrt{\epsilon}}{N^{r+2}}; \tag{3.10a}$$

$$|u_\epsilon - u_{\epsilon,I}|_{\epsilon,G,(1-\lambda,1)} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1}, \qquad |u_\epsilon|_{\epsilon,G,(0,1-\lambda)} \lesssim \frac{1}{N^{r+2}}. \tag{3.10b}$$

Moreover, we can show the following properties.

Lemma 3.3. *Let u_ϵ satisfy the regularity* (2.3). *Then*

$$\sum_{i=N+1}^{2N}\sum_{j=1}^{r}\frac{h_i}{\epsilon}((u_\epsilon - u_{\epsilon,I})(g_{i,j}))^2 \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2(r+1)}, \tag{3.11a}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{r}\frac{h_i}{\epsilon}(u_\epsilon(g_{i,j}))^2 \lesssim \frac{1}{N^{2(r+2)}}. \tag{3.11b}$$

Proof. Let

$$T_1 = \sum_{i=N+1}^{2N}\sum_{j=1}^{r}\frac{h_i}{\epsilon}((u_\epsilon - u_{\epsilon,I})(g_{i,j}))^2, \quad T_2 = \sum_{i=1}^{N}\sum_{j=1}^{r}\frac{h_i}{\epsilon}(u_\epsilon(g_{i,j}))^2.$$

From the standard approximation theory and (2.3), we have

$$\begin{aligned} T_1 &\lesssim \epsilon^{-1}\sum_{i=N+1}^{2N} h_i^{2(r+1)}h_i|u_\epsilon|^2_{r+1,\infty,\tau_i} \\ &\lesssim \epsilon^{-1}\sum_{i=N+1}^{2N}(\frac{h_i}{\epsilon})^{2(r+1)}h_i e^{-2\beta(1-x_i)/\epsilon} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2(r+1)}. \end{aligned}$$

Here we have used the fact that

$$\begin{aligned} h_i e^{-2\beta(1-x_i)/\epsilon} &= h_i e^{-2\beta(1-x_{i-1})/\epsilon} e^{2\beta h_i/\epsilon} \\ &= (N+1)^{\frac{2(r+2)}{N}} h_i e^{-2\beta(1-x_{i-1})/\epsilon} \lesssim \int_{x_{i-1}}^{x_i} e^{-2\beta(1-x)/\epsilon}. \end{aligned}$$

On the other hand, by the regularity (2.3),

$$\begin{aligned} T_2 &\lesssim \epsilon^{-1} \sum_{i=1}^{N} \sum_{j=1}^{r} A_{i,j} e^{-2\beta(1-g_{i,j})/\epsilon} \\ &\lesssim \|e^{-\beta(1-x)/\epsilon}\|^2_{(0,1-\tau)} \lesssim \frac{1}{N^{2(r+2)}}. \end{aligned}$$

Here we have used the remainder for Gaussian quadrature

$$\begin{aligned} &\int_{x_{i-1}}^{x_i} e^{-2\beta(1-x)/\epsilon} - \sum_{j=1}^{r} A_{i,j} e^{-2\beta(1-g_{i,j})/\epsilon} \\ =& \frac{h_i^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3} \left(\frac{2\beta}{\epsilon}\right)^{2r} e^{-2\beta(1-\xi_i)/\epsilon} > 0, \quad \forall i \in \mathbb{Z}_N, \ \ \xi_i \in \tau_i. \end{aligned}$$

The proof is completed. □

We are ready to present our weak continuity of $a_{\mathcal{P}}(\cdot,\cdot)$.

Theorem 3.2. *Let u be the solution of (2.2) and satisfy the regularity (2.3). Let $U^r_{\mathcal{P}}$ be the C^0 finite element space with piecewise polynomials of degree r on the Shishkin mesh. Then*

$$a_{\mathcal{P}}(u - I_\epsilon u, w_{\mathcal{P}'}) \lesssim \left((\frac{\ln(N+1)}{N})^{r+1} + \frac{1}{N^r} \right) \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}. \tag{3.12}$$

Moreover, if $\bar{u} \in U^r_{\mathcal{P}}$, then

$$a_{\mathcal{P}}(u - I_\epsilon u, w_{\mathcal{P}'}) \lesssim \left(\frac{\ln(N+1)}{N} \right)^{r+1} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}. \tag{3.13}$$

Proof. We first estimate the approximation for the regular part. If $\bar{u} \notin U^r_{\mathcal{P}}$, we have, from (2.5) and Cauchy-Schwartz inequality

$$\begin{aligned} a_{\mathcal{P}}(\bar{u} - \bar{u}_I, w_{\mathcal{P}'}) &\lesssim \left(\sum_{i=1}^{2N} \sum_{j=1}^{r} \epsilon h_i (\bar{u} - \bar{u}_I)'(g_{i,j})^2 + \|\bar{u} - \bar{u}_I\|_1^2 \right)^{\frac{1}{2}} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon} \\ &\lesssim \frac{1}{N^r} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}. \end{aligned}$$

Here in the last step, we have used the fact that [34] (p.146, (1.2))

$$|(u - u_I)'(g_{i,j})| \lesssim h^{r+1} |u|_{r+2,\infty,\omega'_{i,j}}, \tag{3.14}$$

with $\omega'_{i,j} = (g_{i,j-1}, g_{i,j+1})$. Furthermore, if $\bar{u} \in U^r_{\mathcal{P}}$, we have $\bar{u} = \bar{u}_I$, which yields

$$a_{\mathcal{P}}(\bar{u} - \bar{u}_I, w_{\mathcal{P}'}) = 0.$$

We next consider the approximation for the singular part. Let

$$K_1 = a_{\mathcal{P}}(u_\epsilon - u_{\epsilon,I}, w_{\mathcal{P}'})_{(1-\lambda,1)}, \quad K_2 = a_{\mathcal{P}}(u_\epsilon, w_{\mathcal{P}'})_{(0,1-\lambda)}.$$

In light of (3.10) and Lemma 3.3, we derive

$$\begin{aligned} K_1 &\lesssim \left(\|u_\epsilon - u_{\epsilon,I}\|^2_{\epsilon,G,(1-\lambda,1)} + \sum_{i=N+1}^{2N}\sum_{j=1}^{r} \frac{h_i}{\epsilon}(u_\epsilon - u_{\epsilon,I})^2(g_{i,j})\right)^{\frac{1}{2}} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon} \\ &\lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}. \end{aligned}$$

Similarly, we derive the following estimate for K_2.

$$K_2 \lesssim \left(\|u_\epsilon\|^2_{\epsilon,G,(0,1-\lambda)} + \sum_{i=1}^{N}\sum_{j=1}^{r} \frac{h_i}{\epsilon}(u_\epsilon(g_{i,j}))^2\right)^{\frac{1}{2}} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon} \lesssim \frac{1}{N^{r+2}}\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}.$$

Recall the bounds of l_λ, we obtain

$$a_{\mathcal{P}}(l_\lambda, w_{\mathcal{P}'}) \lesssim \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}\left(\|l_\lambda\|^2_\epsilon + \|l_\lambda\|^2_1\right)^{\frac{1}{2}} \lesssim \frac{1}{N^{r+1}}\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}.$$

Note that

$$a_{\mathcal{P}}(u_\epsilon - u_{\epsilon,I_\epsilon}, w_{\mathcal{P}'}) = K_1 + K_2 + a_{\mathcal{P}}(l_\lambda, w_{\mathcal{P}'}).$$

Then

$$|a_{\mathcal{P}}(u_\epsilon - u_{\epsilon,I_\epsilon}, w_{\mathcal{P}'})| \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1} \|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}.$$

Combining $a_{\mathcal{P}}(\bar{u} - \bar{u}_I, w_{\mathcal{P}'})$ with $a_{\mathcal{P}}(u_\epsilon - u_{\epsilon,I_\epsilon}, w_{\mathcal{P}'})$, we obtain (3.12) and (3.13). □

3.3. Estimates under the energy norm

In this section, we shall use the inf-sup property (3.8) and the weak continuity (3.12) (or (3.13)) to prove that our finite volume scheme (2.6) has optimal convergence rate under the energy norm.

Lemma 3.4. *Let u satisfy regularity* (2.3)*. Then*

$$\|u - I_\epsilon u\|_\epsilon \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r}. \tag{3.15}$$

Proof. By the approximation theory, there holds

$$\|\bar{u} - \bar{u}_I\|^2_\epsilon \lesssim \left(\frac{\epsilon}{N^{2r}} + \frac{1}{N^{2(r+1)}}\right)\|\bar{u}\|^2_{r+1} \lesssim \frac{1}{N^{2r}}.$$

We next estimate $\|u_\epsilon - u_{\epsilon,I_\epsilon}\|_\epsilon$. By (2.3) and (3.10a),

$$\begin{aligned} \|u_\epsilon - u_{\epsilon,I}\|^2_{\epsilon,(1-\lambda,1)} &\lesssim \sum_{i=N+1}^{2N} \epsilon h_i^{2r}|u_\epsilon|^2_{r+1,\tau_i} + \|u_\epsilon - u_{\epsilon,I}\|^2_{0,(1-\lambda,1)} \\ &\lesssim \left(\epsilon^{-1}\|e^{-\beta(1-x)/\epsilon}\|^2_{0,(1-\lambda,1)} + \epsilon\right)\left(\frac{\ln(N+1)}{N}\right)^{2(r+1)} \\ &\lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}, \end{aligned}$$

$$\|u_\epsilon\|^2_{\epsilon,(0,1-\lambda)} \lesssim \epsilon^{-1}\|e^{-\beta(1-x)/\epsilon}\|^2_{0,(0,1-\lambda)} + \|u_\epsilon\|^2_{0,(0,1-\lambda)} \lesssim \frac{1}{N^{2(r+2)}}.$$

Recall the bounds of l_λ, we derive

$$\|l_\lambda\|_\epsilon^2 = \epsilon |l_\lambda|_1^2 + \|l_\lambda\|_0^2 \lesssim \frac{1}{N^{2r+2}}.$$

Consequently,

$$\|u_\epsilon - u_{\epsilon,I_\epsilon}\|_\epsilon^2 \lesssim \|u_\epsilon - u_{\epsilon,I}\|_{\epsilon,(1-\lambda,1)}^2 + \|u_\epsilon\|_{\epsilon,(0,1-\lambda)}^2 + \|l_\lambda\|_\epsilon^2 \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}.$$

Therefore

$$\|u - I_\epsilon u\|_\epsilon^2 \lesssim \|\bar{u} - \bar{u}_I\|_\epsilon^2 + \|u_\epsilon - u_{\epsilon,I_\epsilon}\|_\epsilon^2 \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}.$$

The inequality (3.15) follows by taking the square roots. □

Theorem 3.3. *Let u and $u_{\mathcal{P}}$ be the solutions of* (2.2) *and* (2.6)*, respectively. If u satisfies the regularity* (2.3)*, then*

$$\|u - u_{\mathcal{P}}\|_\epsilon \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r}. \tag{3.16}$$

Proof. By the inf-sup property (3.8) and the weak continuity (3.12) (or (3.13)),

$$\begin{aligned}\|u_{\mathcal{P}} - I_\epsilon u\|_\epsilon &\lesssim \sup_{w_{\mathcal{P}'} \in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(u_{\mathcal{P}} - I_\epsilon u, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}} \\ &= \sup_{w_{\mathcal{P}'} \in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(u - I_\epsilon u, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r}.\end{aligned}$$

In light of (3.15), we obtain (3.16) immediately. □

Remark 3.1. For reaction-diffusion equations, i.e., the term of first order derivative pu' in (2.1) disappears, the bilinear form $a_{\mathcal{P}}(\cdot,\cdot)$ is uniformly continuous with respect to ϵ. Namely, (3.9) holds. Therefore, we directly have from (3.8) and (3.9)

$$\|u - u_{\mathcal{P}}\|_\epsilon \lesssim \inf_{v_{\mathcal{P}} \in U_{\mathcal{P}}^r} \|u - v_{\mathcal{P}}\|_\epsilon \lesssim \|u - I_\epsilon u\|_\epsilon \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r}.$$

4. Superconvergence

In this section, we present the superconvergence properties of the FVM solution. We begin with a study of superconvergence properties of $u_{\mathcal{P}'}$ at Gauss points.

Theorem 4.1. *Let u be the solution of* (2.2) *and satisfy the regularity* (2.3)*, and $u_{\mathcal{P}}$ the solution of* (2.6)*. Then*

$$\|u - u_{\mathcal{P}}\|_{\epsilon,G} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1} + \frac{1}{N^r}. \tag{4.1}$$

Furthermore, if $\bar{u} \in U_{\mathcal{P}}^r$, then

$$\|u - u_{\mathcal{P}}\|_{\epsilon,G} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1}. \tag{4.2}$$

Proof. We first consider the term $\|u - I_\epsilon u\|_{\epsilon,G}$. By (3.10b) and the bounds for l_λ, we have

$$\begin{aligned}\|u_\epsilon - u_{\epsilon,I_\epsilon}\|_{\epsilon,G} &\le \|u_\epsilon - u_{\epsilon,I}\|_{\epsilon,G,(1-\lambda,1)} + \|u_\epsilon\|_{\epsilon,G,(0,1-\lambda)} + \|l_\lambda\|_{\epsilon,G}\\ &\lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1}.\end{aligned}$$

For the regular part of the solution, we use similar arguments as in Lemma 3.2 to derive

$$\|\bar{u} - \bar{u}_I\|_{\epsilon,G} \lesssim \left(\sum_{i=1}^{2N} h_i h_i^{2(r+1)} \|\bar{u}\|^2_{r+2,\infty,\tau_i} + \|\bar{u} - \bar{u}_I\|_0^2\right)^{\frac{1}{2}} \lesssim \frac{1}{N^{r+1}}.$$

Therefore,

$$\|u - I_\epsilon u\|_{\epsilon,G} \le \|u_\epsilon - u_{\epsilon,I_\epsilon}\|_{\epsilon,G} + \|\bar{u} - \bar{u}_I\|_{\epsilon,G} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1}. \tag{4.3}$$

On the other hand, since

$$\|I_\epsilon u - u_{\mathcal{P}}\|_{\epsilon,G} \sim \|I_\epsilon u - u_{\mathcal{P}}\|_\epsilon.$$

By the inf-sup property (3.8), we have

$$\|u_{\mathcal{P}} - I_\epsilon u\|_{\epsilon,G} \lesssim \sup_{w_{\mathcal{P}'} \in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(u - I_\epsilon u, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}',\epsilon}}.$$

In light of (4.3) and the estimates in Theorem 3.2, the desired results follow from the triangular inequality. □

Since $\|\cdot\|_0 \le \|\cdot\|_{\epsilon,G}$, as a direct consequence of the above theorem, we automatically establish a near optimal convergence rate in the L^2 norm, that is:

$$\|u - u_{\mathcal{P}}\|_0 \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1} + \frac{1}{N^r}, \tag{4.4}$$

and if $\bar{u} \in U^r_{\mathcal{P}}$, then

$$\|u - u_{\mathcal{P}}\|_0 \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1}. \tag{4.5}$$

To discuss superconvergence properties of $u_{\mathcal{P}}$ at nodal points, we need the following assumption

$$\frac{2r+1}{\beta}\epsilon \ln \epsilon^{-1} \le \frac{\overline{h}}{3}. \tag{4.6}$$

Note that (4.6) do not constitute a loss of generality since we are interested in singularly perturbed problems and hence $\epsilon \ll 1$ which makes the assumption very reasonable and it holds even for very large N.

For any $x \in \Omega$, let $G(x,\cdot)$ be the Green function associated with x for the problem (2.2). Then

$$v(x) = A(v, G(x,\cdot)), \quad \forall v \in H_0^1(\Omega),$$

where the Galerkin bilinear form $A(\cdot,\cdot)$ is defined for all $v, w \in H_0^1(\Omega)$ by

$$A(v,w) = \int_0^1 \epsilon a(y) v'(y) w'(y) dy + \int_0^1 ((\epsilon a'(y) + 1) v'(y) + b(y) v(y)) w(y) dy.$$

It is shown in [8] that $G(x,\cdot)$ satisfies the following regularity properties.

$$|G^{(s)}(x,y)| \lesssim (1+\epsilon^{-s}e^{-\beta y/\epsilon}), \qquad \forall y\in(0,x), \tag{4.7a}$$
$$|G^{(s)}(x,y)| \lesssim (1+\epsilon^{-s}e^{-\beta(y-x)/\epsilon}), \quad \forall y\in(x,1), \tag{4.7b}$$

for any $s\geq 0$.

The next theorem provides an error estimate under the L^∞ norm, which plays a critical role in the superconvergence analysis at nodal points.

Theorem 4.2. *If $\bar{u}\in U_{\mathcal{P}}^r$, there holds*

$$\|u-u_{\mathcal{P}}\|_{0,\infty} \lesssim \sqrt{\ln(N+1)}\left(\frac{\ln(N+1)}{N}\right)^{r+\frac{1}{2}}. \tag{4.8}$$

Proof. For any $x\in\tau_i\subset(0,1-\lambda)$, by the inverse inequality,

$$|(I_\epsilon u-u_{\mathcal{P}})(x)| \lesssim \bar{h}^{-\frac{1}{2}}\|I_\epsilon u-u_{\mathcal{P}}\|_{0,\tau_i} \lesssim \bar{h}^{-\frac{1}{2}}\|I_\epsilon u-u_{\mathcal{P}}\|_{\epsilon,G}.$$

For all $x\in(1-\lambda,1)$, we have from Cauchy-Schwartz inequality,

$$|(u_{\mathcal{P}}-I_\epsilon u)(x)| = \left|\int_x^1 (u_{\mathcal{P}}-I_\epsilon u)'(t)dt\right| \leq \sqrt{\frac{\lambda}{\epsilon}}\|I_\epsilon u-u_{\mathcal{P}}\|_{\epsilon,G}.$$

In light of (4.2) and (4.3), we have

$$\|I_\epsilon u-u_{\mathcal{P}}\|_{\epsilon,G} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{r+1},$$

which implies

$$\|I_\epsilon u-u_{\mathcal{P}}\|_{0,\infty} \lesssim \sqrt{\ln(N+1)}\left(\frac{\ln(N+1)}{N}\right)^{r+\frac{1}{2}}.$$

On the other hand, a direct calculation yields

$$\|u-I_\epsilon u\|_{0,\infty} \quad \lesssim \quad \left(\frac{\ln(N+1)}{N}\right)^{r+1}.$$

The desired result (4.8) follows. □

Remark 4.1. In the above theorem, we do not derive an optimal convergence rate for the L^∞ norm, which is of order $r+1$. However, the error bound obtained in Theorem 4.2 is sufficient in our following superconvergence analysis.

With all the preparations , we are ready to present superconvergence properties of $u_{\mathcal{P}}$ at nodal points.

Theorem 4.3. *Let u be the solution of* (2.2) *and satisfy the regularity* (2.3)*, and $u_{\mathcal{P}}$ the solution of* (2.6)*. Assume $\bar{u}\in U_{\mathcal{P}}^r$ and the assumption* (4.6) *holds. Then*

$$|(u-u_{\mathcal{P}})(x_i)| \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}, \quad \forall i\in\mathbb{Z}_{2N}, \tag{4.9}$$

where the hidden constant independent of ϵ and N.

Proof. Let $e = u - u_{\mathcal{P}}$ and

$$V_2(x) = \int_0^x \big((\epsilon a'(y) + 1)e'(y) + b(y)e(y)\big)dy, \quad \forall x \in [0,1].$$

It is shown in [7] that

$$e(x_i) = A(e, G(x_i, \cdot)) = E_1 + E_2, \quad \forall i \in \mathbb{Z}_{2N},$$

where

$$E_1 = \sum_{k=1}^{2N} \frac{h_k^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3} \left[\Big((\epsilon a(y)e'(y) - V_2(y)\Big)\frac{\partial G}{\partial y}(x_i, y)\right]^{(2r)} \Bigg|_{y=\xi_k},$$

$$E_2 = \sum_{k=1}^{2N} \frac{h_k^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3} \left[\frac{\partial G}{\partial y}(x_i, y)\right]^{(2r)} \Bigg|_{y=\eta_k},$$

and $\xi_k, \eta_k \in \tau_k, k \in \mathbb{Z}_{2N}$. When the mesh size $\bar{h}$ is sufficiently small, it is shown in [15] that $\xi_k - x_k \to \frac{h_k}{2}$. Then

$$\xi_k \geq \frac{h_1}{3}, \quad \forall k \in \mathbb{Z}_{2N},$$
$$\xi_k - x_i \geq \xi_k - x_k \geq \frac{h_k}{3}, \quad \forall k \geq i.$$

By the regularity (4.7), for all $j \in \mathbb{Z}_{2r+1}$

$$\begin{aligned}|G^{(j)}(x_i, \xi_k)| &\lesssim \max(1 + \epsilon^{-j}e^{-\beta\xi_k/\epsilon}, 1 + \epsilon^{-j}e^{-\beta(\xi_k - x_i)/\epsilon})\\ &\lesssim \max(1 + \epsilon^{-j}e^{-\beta h_1/3\epsilon}, 1 + \epsilon^{-j}e^{-\beta h_k/3\epsilon}), \quad \forall k \in \mathbb{Z}_{2N}.\end{aligned}$$

In light of (4.6), we have

$$|G^{(j)}(x_i, \xi_k)| \lesssim \begin{cases} 1, & k \in \mathbb{Z}_N, \\ \epsilon^{-j}e^{-\beta h_k/3\epsilon}, & N+1 \leq k \leq 2N. \end{cases} \tag{4.10}$$

We left with the estimate for E_1 and E_2. We only provide details for E_1, since the argument for E_2 is similar (and simpler). We divide E_1 into two parts, outside boundary layer E_1^R and in boundary layer E_1^B. Note that $e^{(j)} = u^{(j)}$ for $j > r$, we have, from the Leibnitz formula of derivative and (4.10)

$$|E_1^R| \lesssim \sum_{k=1}^{N} h_k^{2r+1}\bigg(\sum_{j=0}^{r} |e^{(j)}(\xi_k)| + \sum_{j=r+1}^{2r+1} |u^{(j)}(\xi_k)|\bigg).$$

By the regularity (2.3), there holds for all $j \leq 2r+1$

$$|u_\epsilon^{(j)}(\xi_k)| \lesssim \epsilon^{-j}e^{-\beta(1-\xi_k)/\epsilon} \lesssim \epsilon^{-j}e^{-\beta\bar{h}/3\epsilon} \lesssim 1.$$

Therefore,

$$|u^{(j)}(\xi_k)| \leq |\bar{u}^{(j)}(\xi_k)| + |u_\epsilon^{(j)}(\xi_k)| \lesssim 1.$$

By the inverse inequality, for all $j \in \mathbb{Z}_r$,

$$\begin{aligned}|e^{(j)}(\xi_k)| &\leq \|I_\epsilon u - u_{\mathcal{P}}\|_{j,\infty,\tau_k} + |(I_\epsilon u - u)^{(j)}(\xi_k)|\\ &\lesssim h_k^{-j}\|I_\epsilon u - u_{\mathcal{P}}\|_{0,\infty,\tau_k} + 1.\end{aligned}$$

Consequently,

$$|E_1^R| \lesssim \bar{h}^r \|I_\epsilon u - u_{\mathcal{P}}\|_{0,\infty} + \bar{h}^{2r} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}.$$

Similarly, by (4.8) and (4.10), there goes

$$\begin{aligned}|E_1^B| &\lesssim \sum_{k=N+1}^{2N}\left(\sum_{j=0}^{r}\Big(\frac{h_k}{\epsilon}\Big)^{2r+1-j} h_k^j |e|_{j,\infty,\tau_k} + \sum_{j=r+1}^{2r+1}\Big(\frac{h_k}{\epsilon}\Big)^{2r+1-j} h_k^j |u|_{j,\infty,\tau_k}\right)\\ &\lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r},\end{aligned}$$

where in the last step, we have used

$$h_k|u|_{j,\infty,\tau_k} \lesssim \epsilon^{-j} h_k e^{-\beta(1-x_k)/\epsilon} \lesssim \epsilon^{-j}\int_{x_{k-1}}^{x_k} e^{-\beta(1-x)/\epsilon},\ \forall j,$$

and (see [1])

$$|e|_{j,\infty,\tau_k} \lesssim h_k^{-j}\|e\|_{0,\infty,\tau_k} + h_k^{r+1-j}|u|_{r+1,\infty,\tau_k}, \quad \forall j \in \mathbb{Z}_r.$$

Then

$$|E_1| = |E_1^R + E_1^B| \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}.$$

By the same arguments, we obtain

$$|E_2| \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}.$$

The desired result then follows. $\Box$

As a direct consequence of (4.9), we have

$$E_{Node} = \left(\frac{1}{2N}\sum_{i=1}^{2N}[(u-u_{\mathcal{P}})(x_i)]^2\right)^{\frac{1}{2}} \lesssim \left(\frac{\ln(N+1)}{N}\right)^{2r}. \tag{4.11}$$

5. Numerical Results

In this section, we present numerical examples to support our theoretical findings.

We consider (2.2) with $a=1, b=0$ and $f(x)=x$. The exact solution is

$$u(x) = x\Big(\frac{x}{2}+\epsilon\Big) - \Big(\frac{1}{2}+\epsilon\Big)\Big(\frac{e^{(x-1)/\epsilon}-e^{-1/\epsilon}}{1-e^{-1/\epsilon}}\Big).$$

Note that the regular part of $\bar{u} = x(\frac{x}{2}+\epsilon)$ is included in the trial space $U_{\mathcal{P}}^r, r\geq 2$ and the solution has a boundary layer at $x=1$. The transition point is $1-\lambda$ with $\lambda = \epsilon(r+2)\ln(N+1)$. We solve this problem by the FV scheme (2.6) with $r=3$ and $r=4$, respectively. In our experiments, the underlying meshes are obtained by dividing each interval $(0,1-\lambda)$ and $(1-\lambda,1)$ into $N=2^j$ subintervals, $j\in\mathbb{Z}_8$ when $r=3$ and $j\in\mathbb{Z}_7$ when $r=4$.

Table 5.1: $r = 3$.

	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
N	$\|u-u_{\mathcal{P}}\|_{\epsilon}$	$\|u-u_{\mathcal{P}}\|_{\epsilon,G}$	$\|u-u_{\mathcal{P}}\|_{\epsilon}$	$\|u-u_{\mathcal{P}}\|_{\epsilon,G}$	$\|u-u_{\mathcal{P}}\|_{\epsilon}$	$\|u-u_{\mathcal{P}}\|_{\epsilon,G}$
2	2.2496e-2	8.8874e-3	2.2492e-2	8.8891e-3	2.2492e-2	8.8891e-3
4	1.0257e-2	2.9156e-3	1.0256e-2	2.9157e-3	1.0256e-2	2.9157e-3
8	3.6471e-3	7.2206e-4	3.6464e-3	7.2199e-4	3.6464e-3	7.2199e-4
16	1.0415e-3	1.3485e-4	1.0413e-3	1.3483e-4	1.0412e-3	1.3483e-4
32	2.5206e-4	2.0279e-5	2.5201e-4	2.0275e-5	2.5201e-4	2.0275e-5
64	5.4267e-5	2.6134e-6	5.4256e-5	2.6129e-6	5.4257e-5	2.6129e-6
128	1.0751e-5	3.0171e-7	1.0749e-5	3.0165e-7	1.0750e-5	3.0164e-7
256	2.0038e-6	3.2115e-8	2.0034e-6	3.2108e-8	2.0018e-6	3.2129e-8
	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
N	E_{Node}	$\|u-u_{\mathcal{P}}\|_{L,0}$	E_{Node}	$\|u-u_{\mathcal{P}}\|_{L,0}$	E_{Node}	$\|u-u_{\mathcal{P}}\|_{L,0}$
2	2.5258e-3	2.2876e-3	2.5446e-3	2.3051e-3	2.5448e-3	2.3053e-3
4	2.0226e-4	2.0821e-4	2.0817e-4	2.1305e-4	2.0823e-4	2.1310e-4
8	1.0534e-5	1.8735e-5	1.1693e-5	1.9288e-5	1.1705e-5	1.9294e-5
16	3.8286e-7	1.8008e-6	5.2587e-7	1.8282e-6	5.2786e-7	1.8286e-6
32	1.2225e-8	1.4958e-7	2.0675e-8	1.5019e-7	2.0944e-8	1.5022e-7
64	4.6799e-10	1.0544e-8	7.2865e-10	1.0552e-8	7.6140e-10	1.0574e-8
128	1.6716e-11	6.5709e-10	2.2371e-11	6.5715e-10	2.5781e-11	6.2314e-10
256	5.4219e-13	3.7374e-11	5.8826e-13	3.7002e-11	8.2462e-13	3.0372e-11

We list approximate errors under various (semi-)norms for different values of $\epsilon = 10^{-4}, 10^{-6}$, 10^{-8} in Table 5.1 ($r = 3$) and Table 5.2 ($r = 4$), respectively. Here $|u - u_{\mathcal{P}}|_{L,0}$ denotes an

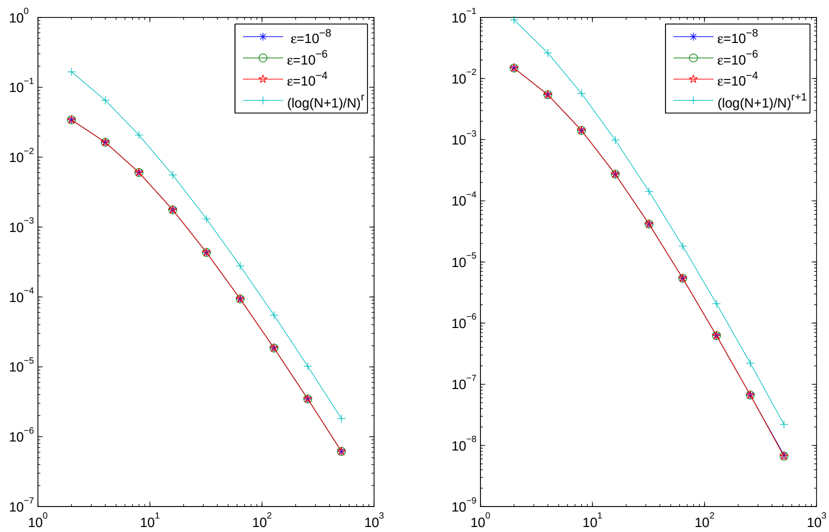

Fig. 5.1. $r = 3$, left: $|u - u_{\mathcal{P}}|_{\epsilon}$, right: $|u - u_{\mathcal{P}}|_{\epsilon,G}$.

Table 5.2: $r = 4$.

	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
N	$\|u - u_{\mathcal{P}}\|_{\epsilon}$	$\|u - u_{\mathcal{P}}\|_{\epsilon,G}$	$\|u - u_{\mathcal{P}}\|_{\epsilon}$	$\|u - u_{\mathcal{P}}\|_{\epsilon,G}$	$\|u - u_{\mathcal{P}}\|_{\epsilon}$	$\|u - u_{\mathcal{P}}\|_{\epsilon,G}$
2	6.7982e-3	2.4927e-3	6.7970e-3	2.4929e-3	6.7970e-3	2.4929e-3
4	2.4190e-3	6.3720e-4	2.4185e-3	6.3714e-4	2.4185e-3	6.3714e-4
8	6.1782e-4	1.1318e-4	6.1764e-4	1.1316e-4	6.1764e-4	1.1316e-4
16	1.1738e-4	1.4022e-5	1.1736e-4	1.4020e-5	1.1736e-4	1.4021e-5
32	1.7806e-5	1.3193e-6	1.7802e-5	1.3191e-6	1.7802e-5	1.3192e-6
64	2.3032e-6	1.0209e-7	2.3029e-6	1.0206e-7	2.3028e-6	1.0107e-7
128	2.6627e-7	6.8753e-9	2.6622e-7	6.8870e-9	2.6618e-7	6.7534e-9
	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
N	E_{Node}	$\|u - u_{\mathcal{P}}\|_{L,0}$	E_{Node}	$\|u - u_{\mathcal{P}}\|_{L,0}$	E_{Node}	$\|u - u_{\mathcal{P}}\|_{L,0}$
2	4.5443e-4	6.4598e-4	4.6189e-4	6.5525e-4	4.6194e-4	6.5535e-4
4	1.9649e-5	3.5114e-5	2.0987e-5	3.6545e-5	2.1001e-5	3.6559e-5
8	4.7100e-7	2.2593e-6	6.0737e-7	2.3259e-6	6.0899e-7	2.3268e-6
16	6.5383e-8	1.5142e-7	1.3277e-8	1.5256e-7	1.3418e-8	1.5257e-7
32	1.003e-10	7.9120e-9	2.4278e-10	7.9218e-9	2.5306e-10	7.8985e-9
64	1.0478e-12	3.3470e-10	3.7264e-12	3.3453e-10	4.3764e-12	3.4772e-10
128	1.6966e-14	1.2164e-11	4.4426e-14	1.2497e-11	5.6245e-14	1.2562e-11

average value of the approximation error at the Lobatto points,

$$|u - u_{\mathcal{P}}|_{L,0} = \left(\frac{1}{2Nr} \sum_{i=1}^{2N} \sum_{j=1}^{r} [u(l_{i,j}) - u_{\mathcal{P}}(l_{i,j})]^2 \right)^{\frac{1}{2}}.$$

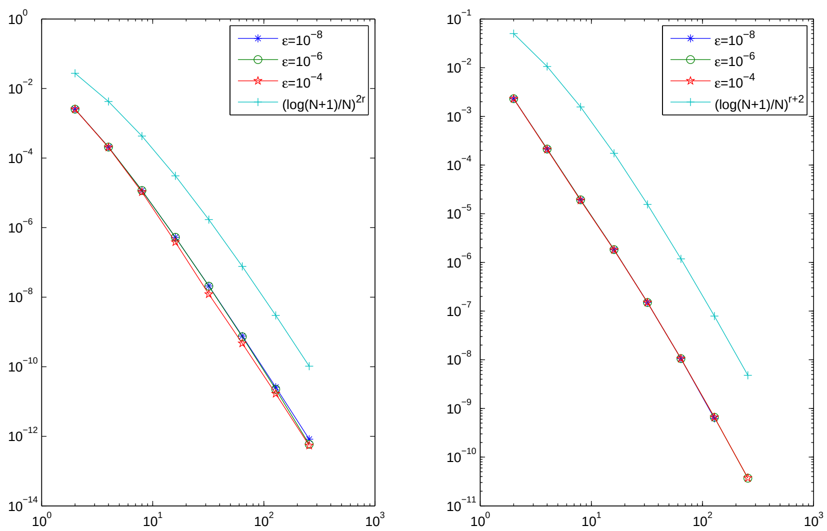

Fig. 5.2. $r = 3$, left: E_{Node}, right: $\|u - u_{\mathcal{P}}\|_{L,0}$.

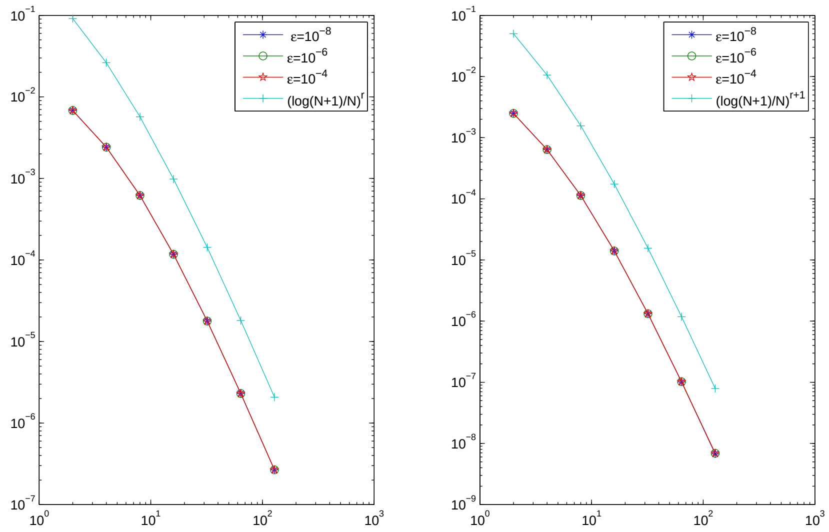

Fig. 5.3. $r = 4$, left: $|u - u_{\mathcal{P}}|_\epsilon$, right: $|u - u_{\mathcal{P}}|_{\epsilon,G}$.

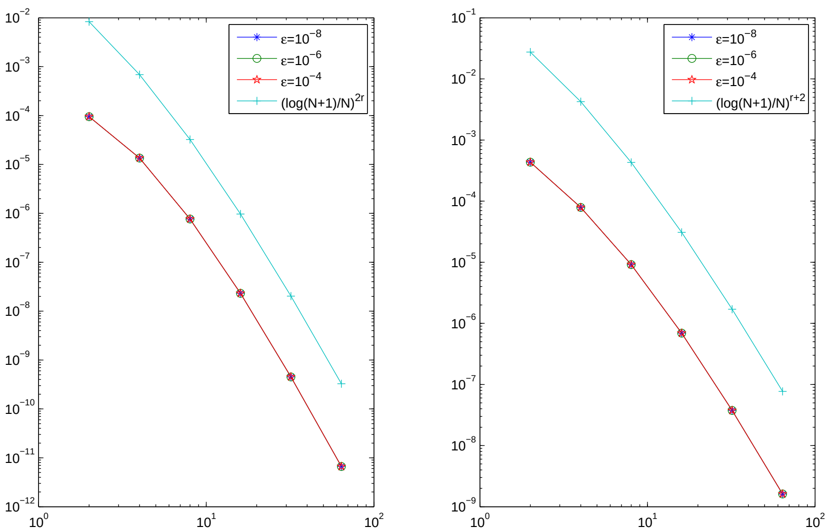

Fig. 5.4. $r = 4$, left: E_{Node}, right: $\|u - u_{\mathcal{P}}\|_{L,0}$.

We may view it as a discrete L^2 norm.

We plot in Figs. 5.1 - 5.4 the convergence curves in various (semi-)norms for different values of $\epsilon = 10^{-4}, 10^{-6}, 10^{-8}$ in cases $r = 3$ and $r = 4$, respectively.

We observe from Figs. 5.1 and 5.3, a near optimal convergence rate $\Big(\ln(N+1)/N\Big)^r$ for $|u - u_{\mathcal{P}}|_\epsilon$ as predicted in Theorem 3.3. We also observe that the error $|u - u_{\mathcal{P}}|_{\epsilon,G}$ decays with an order $\Big(\ln(N+1)/N\Big)^{r+1}$. This confirms the superconvergence results in Theorem 4.1.

The average nodal error E_{Node} is plotted in Figs. 5.2 and 5.4. They clearly indicate a rate of $\Big(\ln(N+1)/N\Big)^{2r}$, which is predicated in Theorem 4.3. Moreover, numerical results show that the logarithmic factor does exist and is not removable. In this sense, the error bound given in Theorem 4.3 is sharp.

We also observe from Figs. 5.2 and 5.4, a rate of $\Big(\ln(N+1)/N\Big)^{r+2}$ for $|u-u_{\mathcal{P}}|_{L,0}$. The error bound here is similar to the counterpart in [7]. This implies that the superconvergence phenomenon at the Lobatto points exists for singularly perturbed problems as well, although its theoretical analysis is lacking at this moment.

Acknowledgments. The second author was supported in part by the US National Science Foundation through grant DMS-1115530, the Ministry of Education of China through the Changjiang Scholars program, and Guangdong Provincial Government of China through the "Computational Science Innovative Research Team" program. The third author was supported in part by the National Natural Science Foundation of China under the grant 11171359 and in part by the Fundamental Research Funds for the Central Universities of China.

References

[1] R.A. Adams and J.F. Fournier, Sobolev Spaces, Singapore press, 1981.

[2] N.S. Bakhvalov, Towards optimization of method for solving boundary value problems in the presence of boundary layers(in Russian), *Zh. Vychisl. Mat. i Mat. Fiz.*, **9** (1969), 841–859.

[3] R.E. Bank and D. J. Rose, Some error estimates for the box scheme. *SIAM J. Numer. Anal.*, **24** (1987), 777–787.

[4] T. Barth and M. Ohlberger, Finite Volume Methods: Foundation and Analysis, Encyclopedia of computational Mechanics, volume 1, chapter 15, John Wiley & Sons, 2004.

[5] Z. Cai, On the finite volume element method, *Numer. Math.*, **58** (1991), 713–735.

[6] Z. Cai and J. Douglas and M. Park, Development and analysis of higher order finite volume methods over rectangles for elliptic equations, *Adv. Comput. Math.*, **19** (2003), 3–33.

[7] W. Cao and Z. Zhang and Q. Zou, Superconvergence of any order finite volume schemes for 1D general elliptic equations, *J. Sci. Comput.*, DOI 10.1007/s10915-013-9696-2, 2013.

[8] F. Celiker and Z. Zhang and H. Zhu, Nodal Superconvergence of SDFEM for Singularly Perturbed problems. *J. Sci. Comput,*, **50** (2012), 405–433.

[9] Z. Chen and J. Wu and Y. Xu, Higher-order finite volume methods for elliptic boundary value problems, *Adv. Comput. Math.*, **37** (2012), 191-253.

[10] P.J. Davis and P. Rabinowitz, Methods of Numerical Integration, 2nd Ed., Academic Press, Boston, 1984.

[11] Ph. Emonot, Methods de Volums Elements Finis: Applications aux Equations de Navier-Stokes et Resultats de Convergence, Lyon, 1992.

[12] R. Ewing and T. Lin and Y. Lin, On the accuracy of the finite volume element based on piecewise linear polynomials, *SIAM J. Numer. Anal.*, **39** (2002), 1865–1888.

[13] R. Eymard and T. Gallouet and R. Herbin, Finite Volume Methods, In: *Handbook of Numerical Analysis*, VII, 713-1020, P.G. Ciarlet and J.L. Lions Eds., North-Holland, Amsterdam, 2000.

[14] R. Li and Z. Chen and W. Wu, The Generalized Difference Methods for Partial Differential Equations, Marcel Dikker, New Youk, 2000.

[15] Y. Li, General theorem on the asymptotic bahavior of the "mid-point" of quadrature remainder, *J. Guizhou Normal University*, **29** (2011), 58-61.

[16] F. Liebau, The finite volume element method with quadratic basis function, *Computing*, **57** (1996), 281–299.

[17] J. Lv and Y. Li, Optimal biquadratic finite volume element methods on quadrilateral meshes, *SIAM. J. Numer. Anal.*, **50** (2012), 2379–2399.

[18] J.J. Miller and E. O'Riordan and G.I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, Word Scientific, Singapore, 1996.

[19] K.W. Morton, Numerical Solution of Convection-Diffusion Problems, Chapman & Hall, London, 1996.

[20] C. Ollivier-Gooch and M. Altena, A high-order-accurate unconstructed mesh finite-volume scheme for the advection-diffusion equation, *J. Comput. Phys.*, **181** (2002), 729–752.

[21] M. Plexousakis and G. Zouraris, On the construction and analysis of high order locally conservative finite volume type methods for one dimensional elliptic problems, *SIAM J. Numer. Anal.*, **42** (2004), 1226–1260.

[22] H.G. Roos and M. Stynes and L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer, Berlin, 1996.

[23] H.G. Roos, Layer-adapter grids for singular pertubation problems, *ZAMMZ. Angew. Math.*, **78** (1998), 291–309.

[24] G.I. Shishkin, Grid Approximation of Singularly Perturbed Ellipric and Parabolic Equations (in Russian), Second doctorial thesis, Keldysh Institute, Moscow, 1990.

[25] M. Stynes, Steady-state convection-diffusion problems, *Acta Numerica.*, **14** (2005), 445–508.

[26] E. Süli, Convergence of finite volume schemes for Poisson's equation on nonuniform meshes, *SIAM J. Numer. Anal.*, **28** (1991), 1419-1430.

[27] T. Wang and Y. Gu, Cubic superconvergence finite volume element method for one-dimensional ellptic and parabolic equations, *J. Comput. Appl. Math.*, **233** (2010), 2285–2301.

[28] Z. Xie and Z. Zhang, Superconvergence of DG method for one-dimensional singularlly perturbed problems, *J. Comput. Math.*, **25** (2007), 185–200.

[29] Z. Xie and Z. Zhang, Uniform superconvergence analysis of the discontinous Galerkin method for a singularly perturbed probelm in 1-D, *Math. Comput.*, **79** (2010), 35–45.

[30] J. Xu and Q. Zou, Analysis of linear and quadratic simplitical finite volume methods for elliptic equations, *Numer. Math.*, **111** (2009), 469–492.

[31] Z. Zhang, Finite Element Superconvergence Approximation for One-Dimensional Singularly Perturbed problems, *Numer. Meth. PDE.*, **18** (2002), 374–395.

[32] Z. Zhang, On the *hp* finite ellment method for the one dimensional singulaly perturbed convection-diffusion problems, *J. Comput. Math.*, **20** (2002), 599-610.

[33] G. Zhou and R. Rannacher, Pointwise superconvergence of the streamline diffusion finite-element method, *Numer. Meth. PDE.*, **12** (1996), 123-145.

[34] Q. Zhu and Q. Lin, Superconvergence Theory of the Finite Element Method (in Chinese), Hunan Science Press, Hunan, 1989.

[35] Q. Zou, Hierarchical error estimates for finite volume approximation solution of elliptic equations, *Appl. Numer. Math.*, **60** (2010), 142–153.