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ROBUST HIGH ORDER CONVERGENCE OF AN OVERLAPPING SCHWARZ METHOD FOR SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEMS*

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Abstract

In this article we propose an overlapping Schwarz domain decomposition method for solving a singularly perturbed semilinear reaction-diffusion problem. The solution to this problem exhibits boundary layers of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$ at both ends of the domain due to the presence of singular perturbation parameter ε. The method splits the domain into three overlapping subdomains, and uses the Numerov or Hermite scheme with a uniform mesh on two boundary layer subdomains and a hybrid scheme with a uniform mesh on the interior subdomain. The numerical approximations obtained from this method are proved to be almost fourth order uniformly convergent (in the maximum norm) with respect to the singular perturbation parameter. Furthermore, it is proved that, for small ε, one iteration is sufficient to achieve almost fourth order uniform convergence. Numerical experiments are given to illustrate the theoretical order of convergence established for the method.

Mathematics subject classification: 65L10, 65L11, 65L20.
Key words: Singular perturbation, Semilinear reaction-diffusion, Overlapping Schwarz method, Robust convergence, Numerov scheme.

1. Introduction

We consider singularly perturbed semilinear reaction-diffusion problem of the form

$$Tu := -\varepsilon u'' + f(x,u) = 0, \quad x \in \Omega = (0,1), \tag{1.1a}$$

$$u(0) = \gamma_0, \quad u(1) = \gamma_1, \tag{1.1b}$$

where $\varepsilon \ll 1$ is a small positive parameter and f is a sufficiently smooth function. In general, as ε tends to zero, the solution u of (1.1) may exhibit boundary and/or internal layers of various types. The location of the layers and the behavior of the solution in these layers depend on the character of f [5,13]. Problems of type (1.1) are probably the most frequently studied singular perturbation problems, both asymptotically and numerically, see [5,9,13,15,16,19,21–23], and the references therein. This interest can be justified by several model problems arising in many areas of science and engineering, such as theory of nonpremixed combustion [24], Michaelis-Menten process in biology [4], and catalytic reaction theory [1].

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We consider problem (1.1) with the assumption that

$$f_u(x,y) \geq \beta > 0 \quad \text{for all } (x,y) \in \Omega \times \mathbb{R}. \tag{1.2}$$

Under this assumption, problem (1.1) and the reduced problem $f(x,u_0(x)) = 0,\ x \in \Omega$, have unique solutions u and u_0 respectively. The solution u generally has exponential boundary layers at $x = 0$ and $x = 1$ of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$. More precisely, u can be decomposed into two parts: $u = v + w$, where for $s = 0,\ldots,6$ and $x \in \overline{\Omega}$ [20]

$$|v^{(s)}(x)| \leq C, \qquad |w^{(s)}(x)| \leq C\varepsilon^{-s/2}\left(e^{(-x\sqrt{\beta/\varepsilon})} + e^{(-(1-x)\sqrt{\beta/\varepsilon})}\right). \tag{1.3}$$

Due to the presence of thin layers, classical numerical methods fail to produce satisfactory results for sigularly perturbed problems, when the perturbation parameter is sufficiently small. This leads to the development of special numerical methods – so called 'parameter-robust' or 'uniformly convergent' numerical methods – that behave uniformly well for all values of the perturbation parameter, no matter how small [7, 17]. Farrell et al. [8] gave a theoretical result which shows that, even in the case of very simple nonlinearity of $f(x,u)$ in u, that is, $f(x,u) = a(u)u$, uniform convergence cannot be achieved in the discrete maximum norm using fitted finite difference schemes on uniform meshes. Most of the works for singularly perturbed semilinear reaction-diffusion problems involve the use of standard finite difference method on special layer-resolving meshes [17]. In the current work we consider a Schwarz domain decomposition method with overlapping subdomains. The origins of this approach can be traced back to the seminal work of Schwarz [18] in the nineteenth century. However, the general development of domain decomposition algorithms occurred only subsequent to the development of parallel computer architectures. More details on domain decomposition can be found in [14].

Suitably designed Schwarz domain decomposition methods have been proven to yield uniformly accurate results for singularly perturbed semilinear reaction-diffusion problems [2, 3, 6]. These methods involve the use of either overlapping subdomains or two overlapping sets of subdomains with no subdomain overlap within each set. The order of uniform convergence in these works is not higher than two. In the current work we design an overlapping Schwarz domain decomposition method that yields almost fourth order uniform approximations for the solution of (1.1). A comprehensive analysis has been given to prove the uniform convergence (in the maximum norm) with respect to the perturbation parameter. Furthermore, we address much faster convergence of the iterative process for small ε. More specifically, it is shown that, for small ε, one iteration is sufficient to achieve almost fourth order uniform convergence.

This paper is organized as follows. In Section 2, we propose an overlapping Schwarz domain decomposition method for solving problem (1.1), and establish nodal error estimates and global error estimates. Results of the numerical experiments are given in Section 3.

Notation. Throughout the paper, C denotes a generic positive constant that is independent of ε, k and the discretization parameter N. We consider the maximum norm and denote it by $||.||_D$, where D is a closed and bounded subset of $\overline{\Omega}$. For a real valued function $g \in C(D)$, we define $||g||_D = \max_{x\in D}|g(x)|$. If $D = \overline{\Omega}$, we drop D from the notation. The analogous discrete maximum norm on the mesh $\overline{\Omega}^N$ is denoted by $||.||_{\overline{\Omega}^N}$. For any functions $g, y_p \in C(\overline{\Omega})$, define $g_i = g(x_i)$, $y_{p,i} = y_p(x_i)$.

2. Overlapping Schwarz Method

2.1. Methodology

We start with decomposition of the domain Ω into three overlapping subdomains $\Omega_p, p = \ell, m, r$:

$$\Omega_\ell = (0, 2\sigma), \qquad \Omega_m = (\sigma, 1-\sigma), \qquad \Omega_r = (1-2\sigma, 1),$$

where the subdomain parameter σ is defined as follows:

$$\sigma = \min\left\{\frac{1}{4}, \quad 4\sqrt{\frac{\varepsilon}{\beta}}\,\ln N\right\}. \tag{2.1}$$

Letting $N = 3 \times 2^j, j \geq 1$, we place a uniform mesh $\overline{\Omega}_p^N = \{x_i\}_{i=0}^N$, with $h_p = x_i - x_{i-1} = (d-a)/N$, on each subdomain $\Omega_p = (a, d)$, $p = l, m, r$. Note that the number of mesh points in each of the three overlapping subdomains need not be equal, but only of the same order, and are considered equal here only to simplify the presentation. The discretization on each subdomain Ω_p^N, $p = \ell, m, r$ is of the form

$$\begin{aligned} T_p^N U_{p,i} := -\varepsilon\delta_x^2 U_{p,i} + q_{p,i}^- f(x_{i-1}, U_{p,i-1}) + q_{p,i}^c f(x_i, U_{p,i}) \\ + q_{p,i}^+ f(x_{i+1}, U_{p,i+1}) = 0, \quad x_i \in \Omega_p^N, \end{aligned} \tag{2.2}$$

where

$$\delta_x^2 U_{p,i} := \frac{1}{h_p^2}(U_{p,i+1} - 2U_{p,i} + U_{p,i-1}),$$

and the coefficients $q_{p,i}^\bullet, \bullet = -, c, +$ are defined as follows. For subdomains Ω_p^N, $p = \ell, r$, the coefficients $q_{p,i}^\bullet$, $i = 1, \ldots, N-1$, $\bullet = -, c, +$ are given by

$$q_{p,i}^- = \frac{1}{12}, \quad q_{p,i}^c = \frac{10}{12}, \quad q_{p,i}^+ = \frac{1}{12}. \tag{2.3}$$

For subdomain Ω_p^N, $p = m$, depending on the relation between h_p and ε, the coefficients $q_{p,i}^\bullet$, $i = 1, \ldots, N-1$, $\bullet = -, c, +$ are defined in two different cases. Let $f_u(x, y) \leq \alpha$, for all $(x, y) \in \overline{\Omega} \times \mathbb{R}$. If $h_p^2\,\alpha \leq 12\varepsilon$, the coefficients $q_{p,i}^\bullet$, $i = 1, \ldots, N-1$, $\bullet = -, c, +$ are defined again by (2.3). For the case $h_p^2\,\alpha > 12\varepsilon$, the coefficients $q_{p,i}^\bullet$, $i = 1, \ldots, N-1$, $\bullet = -, c, +$ are given by

$$q_{p,i}^- = 0, \quad q_{p,i}^c = 1, \quad q_{p,i}^+ = 0. \tag{2.4}$$

Note that, on Ω_ℓ and Ω_r, the Hermite or Numerov scheme is considered; while on Ω_m, a hybrid scheme combining the Hermite scheme and the central scheme is considered.

The iterative process is defined as follows. Choosing initial mesh function $U^{[0]}$ as

$$U^{[0]}(x_i) = u_0(x_i), \quad 0 < x_i < 1, \qquad U^{[0]}(0) = u(0), \quad U^{[0]}(1) = u(1),$$

the mesh function $U^{[k]}$, for all $k \geq 1$, is determined by

$$U^{[k]}(x_i) = \begin{cases} U_\ell^{[k]}(x_i), & x_i \in \overline{\Omega}_\ell^N \setminus \overline{\Omega}_m, \\ U_m^{[k]}(x_i), & x_i \in \overline{\Omega}_m^N, \\ U_r^{[k]}(x_i), & x_i \in \overline{\Omega}_r^N \setminus \overline{\Omega}_m, \end{cases} \tag{2.5}$$

where mesh functions $U_p^{[k]},\ p=\ell,m,r$ satisfy

$$T_\ell^N U_\ell^{[k]}=0 \quad \text{in } \Omega_\ell^N, \qquad U_\ell^{[k]}(0)=u(0), \quad U_\ell^{[k]}(2\sigma)=\mathcal{S}U^{[k-1]}(2\sigma),$$
$$T_r^N U_r^{[k]}=0 \quad \text{in } \Omega_r^N, \qquad U_r^{[k]}(1-2\sigma)=\mathcal{S}U^{[k-1]}(1-2\sigma), \quad U_r^{[k]}(1)=u(1),$$
$$T_m^N U_m^{[k]}=0 \quad \text{in } \Omega_m^N, \qquad U_m^{[k]}(\sigma)=\mathcal{S}U_\ell^{[k]}(\sigma), \quad U_m^{[k]}(1-\sigma)=\mathcal{S}U_r^{[k]}(1-\sigma).$$

Here $\mathcal{S}Z$ denotes the cubic C^0-spline interpolant of Z, obtained by clustering three adjacent and equidistant mesh intervals and fitting a cubic function through the numerical approximation on the four associated mesh points (i.e., on macro intervals $[x_{3i}, x_{3(i+1)}]$, where

$$x_{3i+1}-x_{3i}=x_{3i+2}-x_{3i+1}=x_{3i+3}-x_{3i+2},$$

we fit cubic polynomials). By defining this way the operator $\mathcal{S}$ is uniformly stable [10].

2.2. Nodal error analysis

For convenience of later use, we state the following lemma.

Lemma 2.1. ([11]) *Let u be the solution of* (1.1). *Then*

$$||u|| \le \frac{1}{\beta}||f(.,0)|| + \max\{|\gamma_0|,|\gamma_1|\}.$$

We define $\overline{\Omega}^N := (\overline{\Omega}_\ell^N \backslash \overline{\Omega}_m) \cup \overline{\Omega}_m^N \cup (\overline{\Omega}_r^N \backslash \overline{\Omega}_m)$. In the following theorem we obtain parameter-robust error estimate of the Schwarz iterates.

Theorem 2.1. *Let u be the solution of* (1.1) *and let $U^{[k]}$ be the k^{th} iterate of the discrete Schwarz method of Sect.* 2.1. *Then*

$$||u-U^{[k]}||_{\overline{\Omega}^N} \le C2^{-k} + CN^{-4}\ln^4 N.$$

Proof. Application of the triangle inequality and Lemma 2.1 gives

$$||u-U^{[0]}||_{\Omega^N} \le ||u||_{\Omega^N} + ||u_0||_{\Omega^N} \le C.$$

Also, we have $(u-U^{[k]})(0)=0$ and $(u-U^{[k]})(1)=0$. Then clearly there exist C such that

$$||u-U^{[0]}||_{\overline{\Omega}^N} \le C2^0 + CN^{-4}\ln^4 N.$$

For some $k \ge 0$, suppose

$$||u-U^{[k]}||_{\overline{\Omega}^N} \le C2^{-k} + CN^{-4}\ln^4 N.$$

On the subdomain $\overline{\Omega}_\ell^N$, the error function $\eta_\ell^{[k+1]} = u - U_\ell^{[k+1]}$ solves

$$-\varepsilon\delta_x^2\eta_{\ell,i}^{[k+1]} + q_{\ell,i}^-(f(x_{i-1},u_{i-1}) - f(x_{i-1},U_{\ell,i-1}^{[k+1]})) + q_{\ell,i}^c(f(x_i,u_i) - f(x_i,U_{\ell,i}^{[k+1]}))$$
$$+q_{\ell,i}^+(f(x_{i+1},u_{i+1}) - f(x_{i+1},U_{\ell,i+1}^{[k+1]})) = \varepsilon([Q_\ell u'']_i - \delta_x^2 u_i), \quad x_i \in \Omega_\ell^N,$$

with

$$\eta_\ell^{[k+1]}(0)=0, \qquad \eta_\ell^{[k+1]}(2\sigma) = (u-\mathcal{S}U^{[k]})(2\sigma),$$

where $[Q_p z]_i := q^-_{p,i} z_{i-1} + q^c_{p,i} z_i + q^+_{p,i} z_{i+1}$. The end point 2σ, in general, does not lie in $\overline{\Omega}^N$. Therefore, we use a cubic C^0-spline interpolant of the previous iterate to compute $U_\ell^{[k+1]}(2\sigma)$. Using the solution decomposition $u = v + w$, we get

$$|(u - \mathcal{S}u)(2\sigma)| \leq |(v - \mathcal{S}v)(2\sigma)| + |(w - \mathcal{S}w)(2\sigma)|.$$

Let $\mathcal{I}$ be a cluster of three adjacent mesh intervals of equal length $h_\mathcal{I}$ containing the point 2σ. Then $\mathcal{I} \subset \overline{\Omega}_m$. We make use of the following standard interpolation error estimates

$$|(g - \mathcal{S}g)(2\sigma)| \leq C h_\mathcal{I}^4 ||g^{(4)}||_\mathcal{I} \quad \text{and} \quad |(g - \mathcal{S}g)(2\sigma)| \leq C||g||_\mathcal{I} \quad \text{for } g \in C^4(\mathcal{I}). \tag{2.6}$$

For the regular part v, we use the first bound of (2.6), and (1.3) to get $|(v - \mathcal{S}v)(2\sigma)| \leq CN^{-4}$. While for the layer part w, we consider two distinct cases: $\sigma = 1/4$ and $\sigma < 1/4$. In the first case use the first bound of (2.6) along with (1.3), $\varepsilon^{-1/2} \leq C \ln N$ and $h_\mathcal{I} = \frac{1}{2N}$ to get

$$|(w - \mathcal{S}w)(2\sigma)| \leq CN^{-4} \ln^4 N.$$

The case when $\sigma < 1/4$, use the second bound of (2.6), and (1.3) to get

$$|(w - \mathcal{S}w)(2\sigma)| \leq C2e^{-\sigma\sqrt{\beta/\varepsilon}} \leq CN^{-4}.$$

Using these bounds, the uniform stability of the operator $\mathcal{S}$ [10] and the induction hypothesis, it holds

$$\begin{aligned} |(u - \mathcal{S}U^{[k]})(2\sigma)| &\leq |(u - \mathcal{S}u)(2\sigma)| + |\mathcal{S}(u - U^{[k]})(2\sigma)| \\ &\leq C2^{-k} + CN^{-4} \ln^4 N. \end{aligned} \tag{2.7}$$

For $x_i \in \Omega_\ell^N$, use Taylor expansions, $h_\ell \leq C\sqrt{\varepsilon} N^{-1} \ln N$ and (1.3) to get

$$\varepsilon |([Q_\ell u'']_i - \delta_x^2 u_i)| \leq C\varepsilon h_\ell^4 ||u^{(6)}||_{[x_{i-1}, x_{i+1}]} \leq CN^{-4} \ln^4 N. \tag{2.8}$$

Alternatively, the error equation can be written as

$$-\varepsilon \delta_x^2 \eta_{\ell,i}^{[k+1]} + [Q_\ell (b_\ell \eta_\ell^{[k+1]})]_i = \varepsilon([Q_\ell u'']_i - \delta_x^2 u_i), \quad x_i \in \Omega_\ell^N,$$

where

$$b_{\ell,i} = \int_0^1 f_u(x_i, U_{\ell,i}^{[k+1]} + s(u_i - U_{\ell,i}^{[k+1]}))ds \geq \beta > 0.$$

Introducing the difference operator

$$\mathcal{L}_\ell^N z_i := -\varepsilon \delta_x^2 z_i + [Q_\ell (b_\ell z)]_i, \ x_i \in \Omega_\ell^N,$$

it is easy to see that the matrix associated with $\mathcal{L}_\ell^N$ is an M-matrix. Hence, the difference operator $\mathcal{L}_\ell^N$ satisfy the maximum principle. Recalling (2.7)-(2.8), use the maximum principle for the operator $\mathcal{L}_\ell^N$ with the barrier function $\frac{x_i}{2\sigma} C2^{-k} + CN^{-4} \ln^4 N$, for sufficiently large C to get

$$|\eta_\ell^{[k+1]}(x_i)| \leq \frac{x_i}{2\sigma} C2^{-k} + CN^{-4} \ln^4 N.$$

Consequently

$$||u - U_\ell^{[k+1]}||_{\overline{\Omega}_\ell^N \setminus \overline{\Omega}_m} \leq C2^{-(k+1)} + CN^{-4} \ln^4 N. \tag{2.9}$$

Now use similar arguments to get

$$||u - U_r^{[k+1]}||_{\overline{\Omega}_r^N \setminus \overline{\Omega}_m} \leq C2^{-(k+1)} + CN^{-4}\ln^4 N. \tag{2.10}$$

Thus, we are left to obtain similar bound for $||u - U_m^{[k+1]}||_{\overline{\Omega}_m^N}$. On the subdomain $\overline{\Omega}_m^N$, the error function $\eta_m^{[k+1]} = u - U_m^{[k+1]}$ solves

$$-\varepsilon\delta_x^2\eta_{m,i}^{[k+1]} + [Q_m(b_m\eta_m^{[k+1]})]_i = \varepsilon([Q_m u'']_i - \delta_x^2 u_i), \quad x_i \in \Omega_m^N, \tag{2.11}$$

with

$$\begin{aligned} |\eta_m^{[k+1]}(\sigma)| &= |(u - \mathcal{S}U_l^{[k+1]})(\sigma)| = |(u - U_l^{[k+1]})(\sigma)| \\ &\leq C2^{-(k+1)} + CN^{-4}\ln^4 N, \end{aligned} \tag{2.12}$$

$$\begin{aligned} |\eta_m^{[k+1]}(1-\sigma)| &= |(u - \mathcal{S}U_r^{[k+1]})(1-\sigma)| = |(u - U_r^{[k+1]})(1-\sigma)| \\ &\leq C2^{-(k+1)} + CN^{-4}\ln^4 N, \end{aligned} \tag{2.13}$$

where

$$b_{m,i} = \int_0^1 f_u(x_i, U_{m,i}^{[k+1]} + s(u_i - U_{m,i}^{[k+1]}))ds \geq \beta > 0.$$

Here we have used previous error estimates and the fact that σ and $1-\sigma$, respectively, are the mesh points of Ω_ℓ and Ω_r. We split right-hand side of (2.11) according to the decomposition of u :

$$\varepsilon([Q_m u'']_i - \delta_x^2 u_i) = \varepsilon([Q_m v'']_i - \delta_x^2 v_i) + \varepsilon([Q_m w'']_i - \delta_x^2 w_i). \tag{2.14}$$

Now, to estimate both terms on the right-hand side of (2.14), we consider two distinct cases: $h_m^2\alpha > 12\varepsilon$ and $h_m^2\alpha \leq 12\varepsilon$.

(i) When $h_m^2\alpha > 12\varepsilon$, central differencing is considered. Let $g \in C^4([x_{i-1}, x_{i+1}])$. Then Taylor expansions give

$$|([Q_m g'']_i - \delta_x^2 g_i)| \leq \begin{cases} Ch_m^2||g^{(4)}||_{[x_{i-1},x_{i+1}]}, \\ C||g^{(2)}||_{[x_{i-1},x_{i+1}]}. \end{cases} \tag{2.15}$$

Use the first estimate of (2.15), $h_m \leq CN^{-1}$ and $||v^{(4)}|| \leq C$ to get $\varepsilon|([Q_m v'']_i - \delta_x^2 v_i)| \leq CN^{-4}$. The truncation error for the layer part w is estimated in two distinct cases: $\sigma = 1/4$ and $\sigma < 1/4$. In the first case $\varepsilon^{-1/2} \leq C\ln N$. Thus $||w^{(4)}|| \leq C\ln^4 N$, by (1.3). Using the first bound of (2.15), it follows that

$$\varepsilon|([Q_m w'']_i - \delta_x^2 w_i)| \leq CN^{-4}\ln^4 N.$$

For $\sigma < 1/4$, using the second bound of (2.15), we get

$$\varepsilon|([Q_m w'']_i - \delta_x^2 w_i)| \leq C\varepsilon||w^{(2)}||_{[x_{i-1},x_{i+1}]} \leq CN^{-4}.$$

(ii) When $h_m^2\alpha \leq 12\varepsilon$, the Hermite scheme is considered. Let $g \in C^6([x_{i-1}, x_{i+1}])$. Then Taylor expansions give

$$|([Q_m g'']_i - \delta_x^2 g_i)| \leq \begin{cases} Ch_m^4||g^{(6)}||_{[x_{i-1},x_{i+1}]}, \\ C||g^{(2)}||_{[x_{i-1},x_{i+1}]}. \end{cases} \tag{2.16}$$

The truncation error for the regular part v is bounded using the first estimate of (2.16), $h_m \leq CN^{-1}$ and $||v^{(6)}|| \leq C$. We obtain $\varepsilon|([Q_m v'']_i - \delta_x^2 v_i)| \leq CN^{-4}$. For the other term, we consider two distinct cases: $\sigma = 1/4$ and $\sigma < 1/4$. In the first case, $\varepsilon^{-1/2} \leq C \ln N$. Thus $||w^{(6)}|| \leq C\varepsilon^{-1} \ln^4 N$, by (1.3). Use the first bound of (2.16) to get

$$\varepsilon|([Q_m w'']_i - \delta_x^2 w_i)| \leq CN^{-4} \ln^4 N.$$

While for the other case, use the second bound of (2.16) to get

$$\varepsilon|([Q_m w'']_i - \delta_x^2 w_i)| \leq CN^{-4}.$$

Collecting the various bounds for both the cases, we get

$$\varepsilon|([Q_m u'']_i - \delta_x^2 u_i)| \leq CN^{-4} \ln^4 N, \quad x_i \in \Omega_m^N.$$

Therefore, applying the maximum principle to (2.11)-(2.13) we obtain

$$||u - U_m^{[k+1]}||_{\overline{\Omega}_m^N} \leq C2^{-(k+1)} + CN^{-4} \ln^4 N. \tag{2.17}$$

Combining error bounds (2.9),(2.10) and (2.17) we have the desired result. □

Consider the following discrete problems

$$-\varepsilon\delta_x^2 Z_{p,i} + [Q_p(bZ_p)]_i = 0, \quad x_i \in \Omega_p^N, \quad Z_{p,0} = z_0, \quad Z_{p,N} = z_1, \tag{2.18}$$

and

$$-\varepsilon\delta_x^2 Y_{p,i}^s + [Q_p(\beta Y_p^s)]_i = 0, \quad x_i \in \Omega_p^N, \ s = 1, 2, \tag{2.19a}$$

$$Y_{p,0}^1 = 1, \quad Y_{p,N}^1 = 0, \quad Y_{p,0}^2 = 0, \quad Y_{p,N}^2 = 1, \tag{2.19b}$$

where $b(x_i) \geq \beta$ for $x_i \in \overline{\Omega}_p^N, \beta > 0$ and $p = \ell, m, r$. From the maximum principle, one can easily derive that $0 \leq Y_{p,i}^s \leq 1$, $x_i \in \overline{\Omega}_p^N, s = 1, 2$.

Lemma 2.2. *Suppose that $Z_{p,i}$ and $Y_{p,i}^s$, $s = 1, 2$ are the solutions to the discrete problems* (2.18) *and* (2.19) *respectively. Then*

$$|Z_{p,i}| \leq Y_{p,i}^1 |z_0| + Y_{p,i}^2 |z_1|, \quad x_i \in \overline{\Omega}_p^N.$$

Proof. Suppose that W_p solves

$$-\varepsilon\delta_x^2 W_{p,i} + [Q_p(\beta W_p)]_i = 0, \quad x_i \in \Omega_p^N, \quad W_{p,0} = |z_0|, \quad W_{p,N} = |z_1|.$$

Then W_p can be written as

$$W_{p,i} = Y_{p,i}^1 |z_0| + Y_{p,i}^2 |z_1|, \quad x_i \in \overline{\Omega}_p^N.$$

This can be verified by direct substitution. Using maximum principle, it follows that

$$|Z_{p,i}| \leq W_{p,i}, \quad x_i \in \overline{\Omega}_p^N.$$

This completes the proof. □

The following theorem establishes that the iterative process converges much faster than is shown in Theorem 2.1.

Theorem 2.2. *Let $U^{[k]}$ be the k^{th} iterate of the discrete Schwarz method of Sect.* 2.1. *Then*

$$||U^{[k+1]} - U^{[k]}||_{\overline{\Omega}^N} \leq C\mu^k,$$

where

$$\mu = \left(1 + \frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^{-N} < 1.$$

Furthermore, $\mu \leq 16N^{-4}$ for $\sigma = 4\sqrt{\varepsilon}\ln N/\sqrt{\beta}$.

Proof. The proof is by induction. As $||U^{[1]} - U^{[0]}||_{\Omega^N} \leq ||U^{[1]}||_{\Omega^N} + ||u_0||_{\Omega^N}$, we use a discrete equivalent of Lemma 2.1 in each of the three overlapping subdomains to get

$$||U_\ell^{[1]}||_{\overline{\Omega}_\ell^N} \leq C, \quad ||U_r^{[1]}||_{\overline{\Omega}_r^N} \leq C \quad \text{and} \quad ||U_m^{[1]}||_{\overline{\Omega}_m^N} \leq C.$$

On combining these estimates, it holds

$$||U^{[1]} - U^{[0]}||_{\overline{\Omega}^N} \leq C\mu^0.$$

For some $k \geq 0$, suppose

$$||U^{[k+1]} - U^{[k]}||_{\overline{\Omega}^N} \leq C\mu^k.$$

The mesh function $\xi_\ell^{[k+2]} = U_\ell^{[k+2]} - U_\ell^{[k+1]}$ solves

$$\begin{aligned} &-\varepsilon\delta_x^2\xi_{\ell,i}^{[k+2]} + [Q_\ell(d_\ell\xi_\ell^{[k+2]})]_i = 0, \quad x_i \in \Omega_\ell^N, \\ &\xi_\ell^{[k+2]}(0) = 0, \qquad \xi_\ell^{[k+2]}(2\sigma) = \mathcal{S}(U^{[k+1]} - U^{[k]})(2\sigma), \end{aligned}$$

where

$$d_{\ell,i} = \int_0^1 f_u(x_i, U_{\ell,i}^{[k+1]} + s(U_{\ell,i}^{[k+2]} - U_{\ell,i}^{[k+1]}))ds \geq \beta > 0.$$

By Lemma 2.2 we deduce that

$$|\xi_{\ell,i}^{[k+2]}| \leq Y_{\ell,i}^2|\xi_\ell^{[k+2]}(2\sigma)|, \quad x_i \in \overline{\Omega}_\ell^N.$$

The mesh function Y_ℓ^2 can be written in the form

$$Y_{\ell,i}^2 = \frac{(\zeta_1 + \zeta_2)^i - (\zeta_1 - \zeta_2)^i}{(\zeta_1 + \zeta_2)^N - (\zeta_1 - \zeta_2)^N}, \ i = 0, \ldots, N,$$

where

$$\zeta_1 = \frac{3 + 5\left(\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2}{3 - \left(\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2} \quad \text{and} \quad \zeta_2 = 2\left(\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)\sqrt{\frac{9 + 6\left(\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2}{\left(3 - \left(\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2\right)^2}}.$$

For $x_i \in \Omega_\ell^N \backslash \Omega_m$

$$\begin{aligned} Y_{\ell,i}^2 &\leq \frac{(\zeta_1 + \zeta_2)^{N/2} - (\zeta_1 - \zeta_2)^{N/2}}{(\zeta_1 + \zeta_2)^N - (\zeta_1 - \zeta_2)^N} \\ &\leq \frac{1}{(\zeta_1 + \zeta_2)^{N/2} + (\zeta_1 - \zeta_2)^{N/2}} \leq \frac{1}{(\zeta_1 + \zeta_2)^{N/2}} \leq \mu, \end{aligned}$$

where we have used the inequalities

$$\zeta_1 \geq 1 + \left(\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2 \quad \text{and} \quad \zeta_2 \geq 2\frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}.$$

Using the uniform stability of the operator $\mathcal{S}$ and the induction hypothesis we obtain

$$|\mathcal{S}(U^{[k+1]} - U^{[k]})(2\sigma)| \leq C\mu^k.$$

Hence

$$||U_\ell^{[k+2]} - U_\ell^{[k+1]}||_{\overline{\Omega}_\ell^N \setminus \overline{\Omega}_m} \leq C\mu^{k+1}. \tag{2.20}$$

Similarly we obtain

$$||U_r^{[k+2]} - U_r^{[k+1]}||_{\overline{\Omega}_r^N \setminus \overline{\Omega}_m} \leq C\mu^{k+1}. \tag{2.21}$$

Now, consider the mesh function $\xi_m^{[k+2]} = U_m^{[k+2]} - U_m^{[k+1]}$ satisfying

$$\begin{aligned}
&-\varepsilon\delta_x^2\xi_{m,i}^{[k+2]} + [Q_m(d_m\xi_m^{[k+2]})]_i = 0, \quad x_i \in \Omega_m^N,\\
&\xi_m^{[k+2]}(\sigma) = (U_\ell^{[k+2]} - U_\ell^{[k+1]})(\sigma), \qquad \xi_m^{[k+2]}(1-\sigma) = (U_r^{[k+2]} - U_r^{[k+1]})(1-\sigma),
\end{aligned}$$

where

$$d_{m,i} = \int_0^1 f_u(x_i, U_{m,i}^{[k+1]} + s(U_{m,i}^{[k+2]} - U_{m,i}^{[k+1]}))ds \geq \beta > 0.$$

Applying Lemma 2.2 along with (2.20) and (2.21) we get

$$||U_m^{[k+2]} - U_m^{[k+1]}||_{\overline{\Omega}_m^N} \leq C\mu^{k+1}. \tag{2.22}$$

Combining estimates (2.20),(2.21) and (2.22), we get the desired result.

Now, for $\sigma = 4\sqrt{\varepsilon}\ln N/\sqrt{\beta}$, using arguments given in [12, Lemma 5.1], we obtain

$$\mu = \left(1 + \frac{\sigma}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^{-N} = \left(1 + 4\frac{\ln N}{N}\right)^{-N} \leq 16N^{-4}, \quad N \geq 1.$$

This concludes the proof of the lemma. □

Finally, we combine Theorems 2.1 and 2.2 to prove that, for small ε, only one iteration is required to achieve the desired order of uniform convergence.

Theorem 2.3. *Let u be the solution of* (1.1) *and let $U^{[k]}$ be the k^{th} iterate of the discrete Schwarz method of Sect.* 2.1. *If $\sigma = 4\sqrt{\frac{\varepsilon}{\beta}}\ln N$, then*

$$||u - U^{[k]}||_{\overline{\Omega}^N} \leq C\frac{N^{-4k}}{1 - N^{-4}} + CN^{-4}\ln^4 N.$$

Proof. By Theorem 2.2 there exists U such that $U := \lim_{k\to\infty} U^{[k]}$. Hence, Theorem 2.1 gives

$$||u - U||_{\overline{\Omega}^N} \leq CN^{-4}\ln^4 N. \tag{2.23}$$

From Theorem 2.2 we also have

$$||U^{[k+1]} - U^{[k]}||_{\overline{\Omega}^N} \leq CN^{-4k}.$$

Thus, for $N \geq 2$, it holds

$$||U - U^{[k]}||_{\overline{\Omega}^N} \leq C \sum_{j=k}^{\infty} N^{-4j} = C \frac{N^{-4k}}{1 - N^{-4}}. \tag{2.24}$$

By the triangle inequality and (2.23)-(2.24), we deduce that

$$\begin{aligned} ||u - U^{[k]}||_{\overline{\Omega}^N} &\leq ||U - U^{[k]}||_{\overline{\Omega}^N} + ||u - U||_{\overline{\Omega}^N} \\ &\leq C \frac{N^{-4k}}{1 - N^{-4}} + CN^{-4} \ln^4 N. \end{aligned}$$

This completes the proof. □

2.3. Global error analysis

We extend the nodal parameter-uniform error estimate obtained in the previous section to the global parameter-uniform error estimate.

Theorem 2.4. *Let u be the solution of* (1.1) *and let $U^{[k]}$ be the k^{th} iterate of the discrete Schwarz domain method of Sect.* 2.1. *Then*

$$||u - \mathcal{S}U^{[k]}|| \leq C2^{-k} + CN^{-4} \ln^4 N.$$

Proof. Application of the triangle inequality gives

$$||u - \mathcal{S}U^{[k]}|| \leq ||u - \mathcal{S}u|| + ||\mathcal{S}(u - U^{[k]})||.$$

To bound the second term we use the uniform stability of the operator $\mathcal{S}$ and Theorem 2.1. We get

$$||\mathcal{S}(u - U^{[k]})|| \leq C||u - U^{[k]}||_{\overline{\Omega}^N} \leq C2^{-k} + CN^{-4} \ln^4 N.$$

Next, we bound the first term. Let $\mathcal{I}$ be a cluster of three adjacent mesh intervals of equal length $h_{\mathcal{I}}$. We make use of the following standard interpolation error estimates

$$||g - \mathcal{S}g||_{\mathcal{I}} \leq Ch_{\mathcal{I}}^4 ||g^{(4)}||_{\mathcal{I}}, \quad \text{and} \quad ||g - \mathcal{S}g||_{\mathcal{I}} \leq C||g||_{\mathcal{I}}, \quad \text{for} \quad g \in C^4(\mathcal{I}). \tag{2.25}$$

If $\mathcal{I} \subset (\overline{\Omega}_\ell \backslash \Omega_m) \cup (\overline{\Omega}_r \backslash \Omega_m)$, then $h_{\mathcal{I}} \leq C\sqrt{\varepsilon} N^{-1} \ln N$. Now, use the first estimate of (2.25) along with (1.3) to get

$$||u - \mathcal{S}u||_{\mathcal{I}} \leq CN^{-4} \ln^4 N.$$

If $\mathcal{I} \subset \overline{\Omega}_m$, then we use previous arguments to obtain

$$||u - \mathcal{S}u||_{\mathcal{I}} \leq CN^{-4} \ln^4 N.$$

Theorem 2.5. *Let u be the solution of* (1.1) *and let $U^{[k]}$ be the k^{th} iterate of the discrete Schwarz domain method of Sect.* 2.1. *If $\sigma = 4\sqrt{\frac{\varepsilon}{\beta}} \ln N$, then*

$$||u - \mathcal{S}U^{[k]}|| \leq C \frac{N^{-4k}}{1 - N^{-4}} + CN^{-4} \ln^4 N. \tag{2.26}$$

Proof. Application of the triangle inequality gives

$$||u - \mathcal{S}U^{[k]}|| \leq ||u - \mathcal{S}u|| + ||\mathcal{S}(u - U^{[k]})||.$$

The first term is bounded using arguments in Theorem 2.4. To bound the second term we use the uniform stability of the operator $\mathcal{S}$ and Theorem 2.3. Consequently, we will obtain (2.26). □

3. Numerical Experiments

To demonstrate the theoretical results, we consider the standard test problem from [22]:

$$-\varepsilon u'' + \frac{u-1}{2-u} + g(x) = 0, \qquad u(0) = u(1) = 0,$$

where $g(x)$ is chosen so that the exact solution is

$$u(x) = 1 - \frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}}.$$

A version of this problem, when $g \equiv 0$, represents one of the models for the Michaelis-Menten process in biology [4] and its solution behaves similarly to u above. The stopping criterion for the Schwarz iterations is

$$||U^{[k+1]} - U^{[k]}||_{\overline{\Omega}^N} \leq N^{-4}. \tag{3.1}$$

For the subdomain parameter σ, we take $\beta = 1/4$. We omit the superscript k on the final iterate (when stopping criterion (3.1) is satisfied) and write simply U. For different values of N and ε, we compute maximum nodal errors using $E_\varepsilon^N := ||u - U||_{\overline{\Omega}^N}$. We define a cubic C^0-spline interpolate $\mathcal{S}U$ on macro intervals $[x_{3i}, x_{3(i+1)}]$, $i = 0, \ldots, 2N/3 - 1$, that approximates u on the whole domain. Let $\breve{\overline{\Omega}}^N$ be the mesh that contains the mesh points of the original mesh $\overline{\Omega}^N$ and their midpoints. For different values of N and ε, we take the estimates $\breve{E}_\varepsilon^N := ||u - \mathcal{S}U||_{\breve{\overline{\Omega}}^N}$ for the maximum global errors. Then, we compute $E^N = \max_\varepsilon E_\varepsilon^N$ and $\breve{E}^N = \max_\varepsilon \breve{E}_\varepsilon^N$. Assuming convergence of order $(N^{-1} \ln N)^\varrho$ for some ϱ, we compute the parameter-uniform numerical rates of convergence by

$$\varrho^N = \ln(E^N / E^{2N}) / \ln(2\ln(N)/\ln(2N)),$$

and

$$\breve{\varrho}^N = \ln(\breve{E}^N / \breve{E}^{2N}) / \ln(2\ln(N)/\ln(2N)).$$

Table 3.1: Maximum nodal errors E_ε^N, E^N and uniform convergence rate ϱ^N.

$\varepsilon = 2^{-j}$	$N = 3\times 2^4$	$N = 3\times 2^5$	$N = 3\times 2^6$	$N = 3\times 2^7$	$N = 3\times 2^8$	$N = 3\times 2^9$
$j = 6$	4.75E-08	2.96E-09	1.85E-10	1.18E-11	7.29E-13	4.24E-14
10	1.17E-05	7.36E-07	4.60E-08	2.88E-09	1.80E-10	1.13E-11
14	2.40E-03	1.79E-04	1.17E-05	7.36E-07	4.60E-08	2.88E-09
18	2.40E-03	2.98E-04	3.44E-05	3.60E-06	3.49E-07	3.26E-08
⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	2.40E-03	2.98E-04	3.44E-05	3.60E-06	3.49E-07	3.26E-08
E^N	2.40E-03	2.98E-04	3.44E-05	3.60E-06	3.49E-07	3.26E-08
ϱ^N	3.95	3.91	3.97	4.00	4.00	

We use Newton's method with $u_0 = (1-2g)/(1-g)$ as an initial guess to solve the discrete nonlinear problems involved in our method. In our computations five Newton iterations are sufficient to get the discrete solutions within the tolerance of 10^{-14}. Table 3.1 displays, for different values of ε and N, the maximum nodal error E_ε^N. The last two rows in the table represent the uniform nodal error E^N and the uniform convergence rate ϱ^N. Table 3.2 displays, for different values of ε and N, the maximum global error $\breve{E}_\varepsilon^N$. The last two rows in the table

Table 3.2: Global maximum errors $\breve{E}_\varepsilon^N$, $\breve{E}^N$ and uniform convergence rate $\breve{\varrho}^N$.

$\varepsilon = 2^{-j}$	$N = 3 \times 2^4$	$N = 3 \times 2^5$	$N = 3 \times 2^6$	$N = 3 \times 2^7$	$N = 3 \times 2^8$	$N = 3 \times 2^9$
$j = 6$	1.69E-06	1.11E-07	7.16E-09	4.54E-10	2.85E-11	1.74E-12
10	3.12E-04	2.42E-05	1.69E-06	1.11E-07	7.16E-09	4.54E-10
14	2.57E-02	3.27E-03	3.12E-04	2.42E-05	1.69E-06	1.11E-07
18	2.57E-02	4.93E-03	8.14E-04	1.07E-04	1.19E-05	1.20E-06
⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	2.57E-02	4.93E-03	8.14E-04	1.07E-04	1.19E-05	1.20E-06
$\breve{E}^N$	2.57E-02	4.93E-03	8.14E-04	1.07E-04	1.19E-05	1.20E-06
$\breve{\varrho}^N$	3.12	3.27	3.57	3.75	3.86	

Table 3.3: Number of Schwarz iterations needed to satisfy the stopping criterion.

$\varepsilon = 2^{-j}$	$N = 3 \times 2^4$	$N = 3 \times 2^5$	$N = 3 \times 2^6$	$N = 3 \times 2^7$	$N = 3 \times 2^8$	$N = 3 \times 2^9$
$j = 6$	5	6	7	8	8	9
10	1	1	1	2	2	2
14	1	1	1	1	1	1
18	1	1	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	1	1	1	1	1	1

represent the uniform global error $\breve{E}^N$ and the uniform convergence rate $\breve{\varrho}^N$. Note that the theoretical bounds for both the pointwise error and the global error are asymptotic for large N, that is, they describe what happens only for sufficiently large N. Thus, Table 3.2 is perfectly reasonable, while the better rates of Table 3.1 show only that the asymptotic limit of fourth order convergence is reached earlier (that is, for smaller N) on the original mesh compared with the extended mesh. Table 3.3 displays, for different values of ε and N, iteration counts for our method. One can observe that, for large ε, iteration count increases slightly with N; but for small ε only one Schwarz iteration is required.

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