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ESTIMATIONS OF THE CONSTANTS IN INVERSE INEQUALITIES FOR FINITE ELEMENT FUNCTIONS*

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Abstract

In this paper, we estimate the constants in the inverse inequalities for the finite element functions. Furthermore, we obtain the least upper bounds of the constants in inverse inequalities for the low-order finite element functions. Such explicit estimates of the constants can be used as computable error bounds for the finite element method.

Mathematics subject classification: 65N30.
Key words: Inverse inequality, Finite element function, Bound of the constant.

1. Introduction

As a very effective numerical method of partial differential equations, the finite element method (FEM) is widely applied to the engineering and scientific computation. In the process of analysis and solving by the finite element method, the inverse inequalities are frequently used to bound the high-order (semi-)norms in terms of the low-order ones for the finite element functions, cf. [1, 3]. However, it is well known to prove the inverse inequalities by functional analysis [3], which can not explicitly give the constants on the right and brings troubles into practical error analysis and numerical computation such as a posteriori error estimation and adaptive refinement algorithms. Therefore, for both error analysis and numerical computation, it is very significant to estimate the constants in the inverse inequalities.

We consider the following inverse inequality

$$|v|_{1,\Omega} \leq Ch^{-1}\|v\|_{\Omega}, \tag{1.1}$$

where h is the diameter of the domain Ω and v is a finite element function. The other kinds of inverse inequalities are considered in [4, 5, 7, 10].

For the 1-D and 2-D cases, the constant C in (1.1) is given for the linear finite element in [2]. For any dimension n and order k, The estimation on the constant C in (1.1) is translated into a conditional extremum problem in [9]. However, it needs to solve a system of nonlinear equations with the help of the software Matlab, which is not suitable for theoretic analysis.

In this paper, we explicitly give the inverse inequalities for the finite element functions by different methods. In section 2, due to the recursion relation and orthogonality of Legendre polynomials [8], the constant C in (1.1) is estimated for the 1-D case, which may be extended to general rectangular domains. Especially, we obtained the optimal constants for $k = 1, 2$. In section 3, the constant C in (1.1) is estimated for the reference triangular and tetrahedron finite elements, respectively, which can be ordinarily extended to general bounded domains.

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Furthermore, we obtain the least upper bound of the constant for the linear triangular and tetrahedron finite elements, respectively. In addition, we get an explicit relation between the inverse inequality (1.1) and the geometric characters of the general triangle T. Finally, from the inverse inequality (1.1) we explicitly obtain general inverse inequalities as follows

$$|v|_{m,\Omega} \le Ch^{l-m}|v|_{l,\Omega}, \tag{1.2}$$

where $|v|_{m,\Omega} = \big(\sum\limits_{|\alpha|=m} \frac{|\alpha|!}{\alpha!}\|D^\alpha v\|_\Omega^2\big)^{\frac{1}{2}}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n$ and $m \ge 1$.

2. Inverse Inequalities for the 1-D Case

In this paper, we denote a polynomial space of order $\le k$ on Ω by $P_k(\Omega)$. Let L_k be the k-th Legendre polynomial, that is,

$$L_i(x) = \frac{1}{2^i i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}(x^2-1)^i, \quad i = 0, 1, \dots, k, \tag{2.1}$$

with the following orthogonality

$$(L_i, L_j) \triangleq \int_{-1}^{1} L_i(x)L_j(x)\mathrm{d}x = \begin{cases} 0, & i \ne j, \\ \frac{2}{2i+1}, & i = j. \end{cases} \tag{2.2}$$

For $1 \le i \le k$, according to (2.1) we have the following recursion formula

$$\begin{aligned} L_i'(x) &= \frac{1}{2^i i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}\Big(2ix(x^2-1)^{i-1}\Big) \\ &= \frac{1}{2^{(i-1)}(i-1)!}\frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}\Big((x^2-1)^{i-1} + 2(i-1)x^2(x^2-1)^{i-2}\Big) \\ &= \frac{1}{2^{(i-1)}(i-1)!}\frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}}\Big((2i-1)(x^2-1)^{i-1} + 2(i-1)(x^2-1)^{i-2}\Big) \\ &= (2i-1)L_{i-1}(x) + L_{i-2}'(x). \end{aligned}$$

Then there holds

$$L_i'(x) = \sum_{j=0}^{i-1} d_{ij}L_j(x) = \sum_{j=0}^{k-1} d_{ij}L_j(x), \tag{2.3}$$

where

$$d_{ij} = \begin{cases} 2j+1, & \text{if } i-j \text{ is odd and positive}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

From (2.3) and (2.4), we have the following orthogonality relation

$$(L_{2i}', L_{2j-1}') = 0. \tag{2.5}$$

For any $p(x) \in P_k(-1,1)$, there exist $k+1$ real numbers $c_0, c_1, \dots, c_k$ such that

$$p(x) = \sum_{i=0}^{k} c_i L_i(x), \tag{2.6}$$

$$\|p\|_{L^2(-1,1)}^2 = \Big(\sum_{i=0}^{k} c_i L_i, \sum_{i=0}^{k} c_i L_i\Big) = \sum_{i=0}^{k} c_i^2 (L_i, L_i) = \sum_{i=0}^{k} \frac{2c_i^2}{2i+1}. \tag{2.7}$$

Since

$$p' = \sum_{i=1}^{k} c_i L_i' = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i} L_{2i}' + \sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1} L_{2i-1}',$$

where $\lfloor \frac{k}{2} \rfloor$ denotes the biggest integer smaller than or equal to $\frac{k}{2}$ and $\lceil \frac{k}{2} \rceil$ denotes the smallest integer greater than or equal to $\frac{k}{2}$, according to (2.3)-(2.5) and Hölder inequality we have

$$\begin{aligned}
\|p'\|_{L^2(-1,1)}^2 &= \Big(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i} L_{2i}' + \sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1} L_{2i-1}', \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} c_{2j} L_{2j}' + \sum_{j=1}^{\lceil \frac{k}{2} \rceil} c_{2j-1} L_{2j-1}' \Big) \\
&= \Big(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i} L_{2i}', \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} c_{2j} L_{2j}' \Big) + \Big(\sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1} L_{2i-1}', \sum_{j=1}^{\lceil \frac{k}{2} \rceil} c_{2j-1} L_{2j-1}' \Big) \\
&= \sum_{r=0}^{k-1} \Big(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i} d_{2i,r} \Big)^2 (L_r, L_r) + \sum_{r=0}^{k-1} \Big(\sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1} d_{2i-1,r} \Big)^2 (L_r, L_r) \\
&\le \sum_{r=0}^{k-1} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{d_{2i,r}^2}{(L_{2i}, L_{2i})} (L_r, L_r) \cdot \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i}^2 (L_{2i}, L_{2i}) \\
&\quad + \sum_{r=0}^{k-1} \sum_{i=1}^{\lceil \frac{k}{2} \rceil} \frac{d_{2i-1,r}^2}{(L_{2i-1}, L_{2i-1})} (L_r, L_r) \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1}^2 (L_{2i-1}, L_{2i-1}) \\
&= \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} (4i+1) \sum_{r=0}^{k-1} d_{2i,r} \cdot \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i}^2 (L_{2i}, L_{2i}) + \sum_{i=1}^{\lceil \frac{k}{2} \rceil} (4i-1) \sum_{r=0}^{k-1} d_{2i-1,r} \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1}^2 (L_{2i-1}, L_{2i-1}) \\
&= \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} (4i+1)(2i^2+i) \cdot \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i}^2 (L_{2i}, L_{2i}) + \sum_{i=1}^{\lceil \frac{k}{2} \rceil} (4i-1)(2i^2-i) \cdot \sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1}^2 (L_{2i-1}, L_{2i-1}).
\end{aligned}$$

In the last term above, we have used the following two equalities

$$\sum_{r=0}^{k-1} d_{2i,r} = \sum_{r=0}^{2i-1} d_{2i,r} = 2i^2 + i, \quad \sum_{r=0}^{k-1} d_{2i-1,r} = \sum_{r=0}^{2i-2} d_{2i-1,r} = 2i^2 - i,$$

which follow from (2.4). Define

$$A_k = \max \Big\{ \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} (4i+1)(2i^2+i), \ \sum_{i=1}^{\lceil \frac{k}{2} \rceil} (4i-1)(2i^2-i) \Big\}. \tag{2.8}$$

Then we have

$$\begin{aligned}
\|p'\|_{L^2(-1,1)}^2 &\le A_k \Big(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} c_{2i}^2 (L_{2i}, L_{2i}) + \sum_{i=1}^{\lceil \frac{k}{2} \rceil} c_{2i-1}^2 (L_{2i-1}, L_{2i-1}) \Big) \\
&= A_k \sum_{i=0}^{k} c_i^2 (L_i, L_i) = A_k \|p\|_{L^2(-1,1)}^2.
\end{aligned}$$

It follows from (2.1)-(2.2) and (2.8) that

$$\|L_1'\|_{L^2(-1,1)}^2 = A_1 \|L_1\|_{L^2(-1,1)}^2, \quad \|L_2'\|_{L^2(-1,1)}^2 = A_2 \|L_2\|_{L^2(-1,1)}^2, \tag{2.9}$$

where $A_1 = 3$ and $A_2 = 15$, that is to say, the constant A_k is optimal for $k = 1, 2$.

Therefore, we establish the following lemma.

Lemma 2.1. *For any $p \in P_k(-1, 1)$, there holds*

$$\|p'\|_{L^2(-1,1)} \le \sqrt{A_k}\|p\|_{L^2(-1,1)}, \tag{2.10}$$

where A_k is given in (2.8). *Furthermore, the constant A_k is optimal for $k = 1, 2$.*

Since any open, non-void interval (a, b) may be transformed affinely to $(-1, 1)$ via $x \to -1 + 2\frac{x-a}{b-a}$, from Lemma 2.1 we obtain

Theorem 2.1. *For any $p \in P_k(a, b)$, there holds*

$$\|p'\|_{L^2(a,b)} \le \frac{C_{1,k}}{h}\|p\|_{L^2(a,b)}, \tag{2.11}$$

where $h = b - a, C_{1,k} = 2\sqrt{A_k}$ and A_k is given in (2.8). *Furthermore, the constant in* (2.11) *is optimal for $k = 1, 2$.*

Remark 2.1. According to Theorem 2.1, for $n = k = 1$ we have $A_1 = 3$ and $C_{1,1} = 2\sqrt{3}$ in (1.1), which is the same as that in [2].

Remark 2.2. By using Fubini's theorem and Theorem 2.1, we can easily obtain the inverse inequalities on rectangular domains as follows.

Theorem 2.2. *Assume $K = (a_1, b_1)\times(a_2, b_2)\times\ldots\times(a_n, b_n)$ and $Q_k(K)$ denotes the space of all polynomials that are of degree $\le k$ with respect to each variable x_i. Then for any $v \in Q_k(K)$,we have*

$$|v|_{1,K} \le \frac{C_{1,k}}{h_K}\|v\|_K, \tag{2.12}$$

and especially the anisotropic inverse inequalities

$$\|\frac{\partial v}{\partial x_i}\|_{0,K} \le \frac{C_{1,K}}{h_i}\|v\|_K, \quad 1 \le i \le n, \tag{2.13}$$

where $h_K = (\sum_{i=1}^n \frac{1}{(b_i-a_i)^2})^{-\frac{1}{2}},\ h_i = b_i - a_i,\ C_{1,k} = 2\sqrt{A_k}$ and A_k is given in (2.8). *Furthermore, the constants in* (2.12)-(2.13) *are optimal for $k = 1, 2$.*

3. Inverse Inequalities on the Simplex

The orthogonal polynomials on the triangle or tetrahedron can not be easily be expressed like that for 1-D case. To this end, we will introduce another method in the section to avoid the orthogonalization.

3.1. Inverse inequalities on the triangle

Let h, a and θ be positive constants such that

$$h > 0, \quad 0 < a \le 1, \quad \frac{\pi}{3} \le \cos^{-1}\frac{a}{2} \le \theta < \pi. \tag{3.1}$$

We define the triangle $T_{a,\theta,h}$ by $\triangle OAB$ with three vertices $O(0,0)$, $A(h,0)$ and $B(ah\cos\theta, ah\sin\theta)$ as that in [6]. From (3.1), $\angle BOA = \theta$ is the maximum interior angle and AB the edge of maximum length, i.e., $|AB| \geq h \geq ah$, so that $h = |OA|$ here denotes the medium edge length, although the notation h is often used as the largest edge length. Since we can configure any triangle T as $T_{a,\theta,h}$ by an appropriate congruent transformation in R^2, then we regard $T_{a,\theta,h}$ as a general triangle T. We will use abbreviated notations $T = T_{a,\theta,h}$ and $\hat{T} = T_{1,\frac{\pi}{2},1}$, as illustrated in Fig. 3.1.

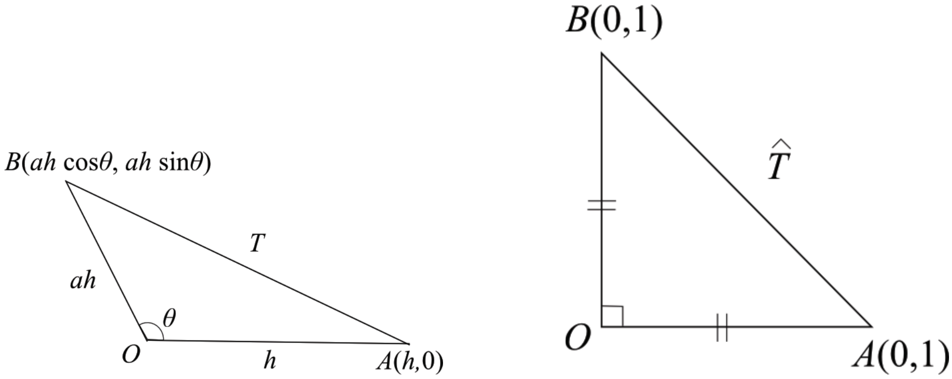

Fig. 3.1. Notations for triangles: $T = T_{a,\theta,h}$, $\hat{T} = T_{1,\frac{\pi}{2},1}$.

Let $N = \dim P_k(\hat{T})$ and $\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N\}$ be a given basis of $P_k(\hat{T})$ such that $\|\hat{p}_i\|_{\hat{T}} = 1$, there must exist the corresponding basis $\{\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_N\}$ in $P_k(\hat{T})$ such that

$$\hat{v} = \sum_{i=1}^{N} \Big(\int_{\hat{T}} \hat{v}\hat{p}_i \mathrm{d}\hat{x} \Big) \hat{\varphi}_i, \quad \forall \hat{v} \in P_k(\hat{T}), \tag{3.2}$$

where $\int_{\hat{T}} \hat{p}_i \hat{\varphi}_j \mathrm{d}\hat{x} = \delta_{ij}$. According to Minkowski's inequality and Hölder's inequality, we have

$$\|\hat{v}_{\hat{x}_j}\|_{\hat{T}} \leq \sum_{i=1}^{N} \Big| \int_{\hat{T}} \hat{v}\hat{p}_i \mathrm{d}\hat{x} \Big| \cdot \Big\| \frac{\partial \hat{\varphi}_i}{\partial \hat{x}_j} \Big\|_{\hat{T}} \leq \Big(\sum_{i=1}^{N} \| \frac{\partial \hat{\varphi}_i}{\partial \hat{x}_j} \|_{\hat{T}} \Big) \|\hat{v}\|_{\hat{T}}, \quad j = 1, 2. \tag{3.3}$$

Define

$$\hat{B}_k = \sum_{j=1}^{2} \Big(\sum_{i=1}^{N} \| \frac{\partial \hat{\varphi}_i}{\partial \hat{x}_j} \|_{\hat{T}} \Big)^2. \tag{3.4}$$

Then there holds

$$|\hat{v}|_{1,\hat{T}}^2 \leq \hat{B}_k \|\hat{v}\|_{\hat{T}}^2. \tag{3.5}$$

This establishes the following lemma.

Lemma 3.1.

$$|\hat{v}|_{1,\hat{T}} \leq \sqrt{\hat{B}_k} \|\hat{v}\|_{\hat{T}}, \quad \forall \hat{v} \in P_k(\hat{T}), \tag{3.6}$$

where $\hat{B}_k$ *is given in* (3.4).

The inverse inequality (3.6) is a general one. Now for practical application we give sharper estimates of the constants for $k = 1, 2$, respectively.

To this end, let $\hat{p}_1=\sqrt{2},\ \hat{p}_2=\sqrt{12}(\hat\lambda_2-\hat\lambda_3),\ \hat{p}_3=2(3\hat\lambda_1-1),\ \hat{p}_4=\sqrt{6}(1-8\hat\lambda_1+10\hat\lambda_1^2),\ \hat{p}_5=\sqrt{18}(4\hat\lambda_2-4\hat\lambda_3-5\hat\lambda_2^2+5\hat\lambda_3^2)$ and $\hat{p}_6=\sqrt{45}(\hat\lambda_2^2-4\hat\lambda_2\hat\lambda_3+\hat\lambda_3^2)$. Here $\hat\lambda_1=1-\hat{x}_1-\hat{x}_2,\ \hat\lambda_2=\hat{x}_1$ and $\hat\lambda_3=\hat{x}_2$ are the area coordinates of $\hat{T}$.

By simple computations we have

$$\int_{\hat{T}}\hat{p}_i\hat{p}_j\mathrm{d}\hat{x}=\delta_{ij},\quad 1\le i,j\le 6,$$

which means that $\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}$ and $\{\hat{p}_1,\hat{p}_2,\dots,\hat{p}_6\}$ are the L^2-orthogonal bases of $P_1(\hat{T})$ and $P_2(\hat{T})$, respectively.

(*i*) For any $\hat{v}\in P_1(\hat{T})$, we have

$$\hat{v}=a_1\hat{p}_1+a_2\hat{p}_2+a_2\hat{p}_3,\quad \|\hat{v}\|_{\hat{T}}^2=a_1^2+a_2^2+a_3^2,$$

where $a_i=\int_{\hat{T}}\hat{v}\hat{p}_i\mathrm{d}\hat{x}$ for $i=1,2,3$. Then there holds

$$\begin{aligned}|\hat{v}|_{1,\hat{T}}^2&=\int_{\hat{T}}\Big(\Big(\frac{\partial\hat{v}}{\partial\hat{x}_1}\Big)^2+\Big(\frac{\partial\hat{v}}{\partial\hat{x}_2}\Big)^2\Big)\mathrm{d}\hat{x}\\&=\Big(\sqrt{12}a_2-6a_3)^2+(-\sqrt{12}a_2-6a_3)^2\Big|\hat{T}|\\&=12a_2^2+36a_3^2\le 36(a_1^2+a_2^2+a_3^2)=36\|\hat{v}\|_{\hat{T}}^2.\end{aligned}$$

Hence for $k=1$ we can set $\hat{B}_1=36$ in (3.6) that is

$$|\hat{v}|_{1,\hat{T}}\le 6\|\hat{v}\|_{\hat{T}},\quad \forall\hat{v}\in P_1(\hat{T}).\tag{3.7}$$

Furthermore, the above constant is optimal, since $|\hat{p}_3|_{1,\hat{T}}=6\|\hat{p}_3\|_{\hat{T}}$.

Remark 3.1. In [2, 9] the following inverse inequality is given

$$|\hat{v}|_{1,\hat{T}}\le 6\sqrt{2}\|\hat{v}\|_{\hat{T}},\quad \forall\hat{v}\in P_1(\hat{T}).$$

Therefore, we give the better result on the constant for $k=1$.

(*ii*) For any $\hat{v}\in P_2(\hat{T})$, similarly we have

$$\hat{v}=\sum_{i=1}^6 a_i\hat{p}_i,\quad \|\hat{v}\|_{\hat{T}}^2=\sum_{i=1}^6 a_i^2,$$

where $a_i=\int_{\hat{T}}\hat{v}\hat{p}_i\mathrm{d}\hat{x}$ for $i=1,2,\dots,6$. Then there holds

$$|\hat{v}|_{1,\hat{T}}^2=\sum_{i=1}^6\sum_{j=1}^6 a_ia_jd_{ij},\tag{3.8}$$

where $d_{ij}=\int_{\hat{T}}(\frac{\partial\hat{p}_i}{\partial\hat{x}_1}\frac{\partial\hat{p}_j}{\partial\hat{x}_1}+\frac{\partial\hat{p}_i}{\partial\hat{x}_2}\frac{\partial\hat{p}_j}{\partial\hat{x}_2})\mathrm{d}\hat{x}$. By some computations we have

$$\begin{aligned}&d_{23}=d_{24}=d_{26}=d_{35}=d_{45}=d_{56}=0,\\&d_{22}=12,\quad d_{33}=36,\quad d_{44}=144,\quad d_{55}=108,\quad d_{66}=90,\\&d_{25}=4\sqrt{6},\quad d_{34}=-8\sqrt{6},\quad d_{36}=12\sqrt{5},\quad d_{46}=-6\sqrt{30},\\&a_2a_5d_{25}\le 24a_2^2+a_5^2,\quad a_3a_4d_{36}\le 84a_3^2+\frac{8}{7}a_4^2,\\&a_3a_6d_{36}\le 30a_3^2+6a_6^2,\quad a_4a_6d_{46}\le 5a_4^2+54a_6^2.\end{aligned}$$

Substituting the above results into (3.8), we get

$$|\hat{v}|^2_{1,\hat{T}} \le 36a_2^2 + 150a_3^2 + \Big(150 + \frac{1}{7}\Big)a_4^2 + 109a_5^2 + 150a_6^2 \le \Big(150 + \frac{1}{7}\Big)\|\hat{v}\|^2_{\hat{T}}.$$

Hence for $k = 2$ we can set $\hat{B}_2 = 150 + \frac{1}{7}$ in (3.6). That is

$$|\hat{v}|_{1,\hat{T}} \le \sqrt{150 + \frac{1}{7}}\|\hat{v}\|_{\hat{T}}, \quad \forall \hat{v} \in P_2(\hat{T}). \tag{3.9}$$

Remark 3.2. In [9] the following inverse inequality is given

$$|\hat{v}|_{1,\hat{T}} \le 17.7246\|\hat{v}\|_{\hat{T}}, \quad \forall \hat{v} \in P_2(\hat{T}).$$

Therefore, we also give the better result on the constant for $k = 2$.

Next we start to explicitly establish inverse inequalities on the general triangle T that is $T_{a,\theta,h}$. Let us introduce the following simple affine transformation $F : \hat{T} \to T_{a,\theta,h}$ by

$$x = F(\hat{x}) = B_{a,\theta,h}\hat{x}, \tag{3.10}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}, \quad B_{a,\theta,h} = \begin{pmatrix} h & ah\cos\theta \\ 0 & ah\sin\theta \end{pmatrix}.$$

Theorem 3.1. *The following estimate holds:*

$$|v|_{1,T} \le \frac{C_{2,k}}{h}\|v\|_T, \quad \forall v \in P_k(T), \tag{3.11}$$

where $C_{2,k} = \frac{\sqrt{(1+|\cos\theta|)\hat{B}_k}}{a\sin\theta}$ *and* $\hat{B}_k$ *is given in* (3.4).

Proof. Consider the inverse affine transformation $F^{-1} : T_{a,\theta,h} \to \hat{T}$ as follows

$$\hat{x} = F^{-1}(x) = B^{-1}_{a,\theta,h}x,$$

where

$$B^{-1}_{a,\theta,h} = \frac{1}{ah\sin\theta}\begin{pmatrix} a\sin\theta & -a\cos\theta \\ 0 & 1 \end{pmatrix}.$$

By simple calculations, we have for $\hat{v} = v \circ F$ under the above transformation.

$$\begin{aligned}\sum_{i=1}^{2}\Big(\frac{\partial v}{\partial x_i}\Big)^2 &= \frac{1}{h^2\sin^2\theta}\Big(\Big(\frac{\partial \hat{v}}{\partial \hat{x}_1}\Big)^2 - \frac{2\cos\theta}{a}\frac{\partial \hat{v}}{\partial \hat{x}_1}\cdot\frac{\partial \hat{v}}{\partial \hat{x}_2} + \frac{1}{a^2}\Big(\frac{\partial \hat{v}}{\partial \hat{x}_2}\Big)^2\Big) \\ &\le \frac{1+|\cos\theta|}{a^2h^2\sin^2\theta}\sum_{i=1}^{2}\Big(\frac{\partial \hat{v}}{\partial \hat{x}_i}\Big)^2.\end{aligned}$$

According to Lemma 3.1, we have

$$\begin{aligned}|v|^2_{1,T} &\le \frac{1+|\cos\theta|}{a^2h^2\sin^2\theta}|\det B_{a,\theta,h}||\hat{v}|^2_{1,\hat{T}} \\ &\le \frac{1+|\cos\theta|}{a^2h^2\sin^2\theta}\hat{B}_k|\det B_{a,\theta,h}|\|\hat{v}\|^2_{\hat{T}} \\ &= \frac{1+|\cos\theta|}{a^2h^2\sin^2\theta}\hat{B}_k\|v\|^2_T.\end{aligned}$$

Let $C_{2,k} = \frac{\sqrt{(1+|\cos\theta|)\hat{B}_k}}{a\sin\theta}$ and the proof is complete. ☐

3.2. Inverse inequalities on the tetrahedron

Let $\hat{T}$ be the reference tetrahedron with vertices $\hat{b}_0(0,0,0), \hat{b}_1(1,0,0), \hat{b}_2(0,1,0)$ and $\hat{b}_3(0,0,1)$, then $\hat{\lambda}_0 = 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3$ and $\hat{\lambda}_i = \hat{x}_i$ $(i = 1,2,3)$ are the volume coordinates of $\hat{T}$. Similar to the triangular case, let $N = \dim P_k(\hat{T})$ and $\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N\}$ be a given basis of $P_k(\hat{T})$ such that $\|\hat{p}_i\|_{\hat{T}} = 1$, there must exist the corresponding basis $\{\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_N\}$ in $P_k(\hat{T})$ such that

$$\hat{v} = \sum_{i=1}^{N} \Big(\int_{\hat{T}} \hat{v}\hat{p}_i \mathrm{d}\hat{x} \Big) \hat{\varphi}_i, \quad \forall \hat{v} \in P_k(\hat{T}), \tag{3.12}$$

where $\int_{\hat{T}} \hat{p}_i \hat{\varphi}_j \mathrm{d}\hat{x} = \delta_{ij}$.

In the same way as the triangular case, we have

Lemma 3.2. *For the reference tetrahedron $\hat{T}$, we have*

$$|\hat{v}|_{1,\hat{T}} \le \sqrt{\hat{B}_k} \|\hat{v}\|_{\hat{T}}, \quad \forall \hat{v} \in P_k(\hat{T}), \tag{3.13}$$

where

$$\hat{B}_k = \sum_{j=1}^{3} \Big(\sum_{i=1}^{N} \| \frac{\partial \hat{\varphi}_i}{\partial x_j} \|_{\hat{T}} \Big)^2. \tag{3.14}$$

Likewise for practical application we give sharper estimate of the constant for $k = 1$. Let $\hat{p}_0 = \sqrt{6}$, $\hat{p}_1 = \sqrt{60}(\hat{\lambda}_0 - \hat{\lambda}_1)$, $\hat{p}_2 = \sqrt{60}(\hat{\lambda}_2 - \hat{\lambda}_3)$ and $\hat{p}_3 = \sqrt{120}(\hat{\lambda}_2 + \hat{\lambda}_3 - \frac{1}{2})$, then it is easy to get

$$\int_{\hat{T}} \hat{p}_i \hat{p}_j \mathrm{d}\hat{x} = \delta_{ij}, \quad 0 \le i, j \le 3,$$

that is to say, $\{\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3\}$ is the L^2-orthogonal base of $P_1(\hat{T})$.

For any $\hat{v} \in P_1(\hat{T})$, we define

$$\hat{v} = \sum_{i=0}^{3} a_i \hat{p}_i, \quad \|\hat{v}\|^2_{\hat{T}} = \sum_{i=0}^{3} a_i^2,$$

where $a_i = \int_{\hat{T}} \hat{v}\hat{p}_i \mathrm{d}\hat{x}$ for $i = 0,1,2,3$. Then there holds

$$\begin{aligned}
|\hat{v}|^2_{1,\hat{T}} &= \int_{\hat{T}} \sum_{j=1}^{3} \Big(\frac{\partial \hat{v}}{\partial \hat{x}_j} \Big)^2 \mathrm{d}\hat{x} \\
&= |\hat{T}| \Big(240a_1^2 + 60(\sqrt{2}a_3 - a_1 + a_2)^2 + 60(\sqrt{2}a_3 - a_1 - a_2)^2 \Big) \\
&= 40a_1^2 + 20(\sqrt{2}a_3 - a_1)^2 + 20a_2^2 \\
&\le 40a_1^2 + 20\Big(2a_3^2 + (a_1^2 + 2a_3^2) + a_1^2 \Big) + 20a_2^2 \\
&\le 80 \sum_{i=0}^{3} a_i^2 = 80 \|\hat{v}\|^2_{\hat{T}}.
\end{aligned}$$

Hence for the linear tetrahedral element we can set $\hat{B}_1 = 80$ in (3.13), that is

$$|\hat{v}|_{1,\hat{T}} \le 4\sqrt{5} \|\hat{v}\|_{\hat{T}}, \quad \forall v \in P_1(\hat{T}). \tag{3.15}$$

Furthermore, the above constant is optimal. In fact, for $\hat{v} = -\sqrt{2}\hat{p}_1 + \hat{p}_3$ there holds $|\hat{v}|_{1,\hat{T}} = 4\sqrt{5}\|\hat{v}\|_{\hat{T}}$.

In the similar way as the triangular case, we can get the inverse inequalities on a general tetrahedron.

Remark 3.3. Obviously, the above method for the complex can be generalized to any bounded convex domain and get corresponding inverse inequalities for the polynomial spaces.

4. General Inverse Inequalities

In this section, we discuss general inverse inequalities from Theorems 2.1 and 3.1 for the 1-D and 2-D cases.

Theorem 4.1. *Assume $0 \le l < m$, T is an interval (a,b) for 1-D case and a triangle for 2-D case, then for any $v \in P_k(T)$, there holds*

$$|v|_{m,T} \le \frac{D_{n,ml}}{h^{m-l}}|v|_{l,T}, \tag{4.1}$$

where $D_{n,ml} = C_{n,k-l}C_{n,k-l-1}\dots C_{n,k-m+1}$, $h = b-a$ for 1-D case, h is the medium edge length and $C_{n,i}$ is respectively given in Theorems 2.1 and 3.1 for $n = 1,2$.

Proof. We only prove (4.1) for $n = 2$. For $n = 1$, according to Theorem 2.1 we can easily get (4.1) by analogy.

Since $D^\alpha v \in P_{k-|\alpha|}(T)$, according to Theorem 3.1 we have

$$
\begin{aligned}
|v|^2_{l+1,T} &= \sum_{|\alpha|=l+1} \frac{(l+1)!}{\alpha!}\|D^\alpha v\|^2_T \\
&= \sum_{\substack{|\alpha|=l+1\\ \alpha_1>0,\alpha_2>0}} \frac{l!(\alpha_1+\alpha_2)}{\alpha!}\|D^\alpha v\|^2_T + \|D^{(l+1,0)}v\|^2_T + \|D^{(0,l+1)}v\|^2_T \\
&= \sum_{\substack{|\alpha|=l+1\\ \alpha_1>0,\alpha_2>0}} \Big(\frac{l!}{(\alpha_1-1)!\alpha_2!} + \frac{l!}{\alpha_1!(\alpha_2-1)!}\Big)\|D^\alpha v\|^2_T + \|D^{(l+1,0)}v\|^2_T + \|D^{(0,l+1)}v\|^2_T \\
&= \sum_{\substack{|\alpha|=l+1\\ \alpha_1>0}} \frac{l!}{(\alpha_1-1)!\alpha_2!}\|D^\alpha v\|^2_T + \sum_{\substack{|\alpha|=l+1\\ \alpha_2>0}} \frac{l!}{\alpha_1!(\alpha_2-1)!}\|D^\alpha v\|^2_T \\
&= \sum_{|\alpha|=l} \frac{l!}{\alpha!}\|D^{\alpha+(1,0)}v\|^2_T + \sum_{|\alpha|=l} \frac{l!}{\alpha!}\|D^{\alpha+(0,1)}v\|^2_T \\
&= \sum_{|\alpha|=l} \frac{l!}{\alpha!}\Big(\|D^{\alpha+(1,0)}v\|^2_T + \|D^{\alpha+(0,1)}v\|^2_T\Big) = \sum_{|\alpha|=l} \frac{l!}{\alpha!}|D^\alpha v|^2_{1,T} \\
&\le \Big(\frac{C_{2,k-l}}{h}\Big)^2 \sum_{|\alpha|=l} \frac{l!}{\alpha!}\|D^\alpha v\|^2_T = \Big(\frac{C_{2,k-l}}{h}\Big)^2 |v|^2_{l,T}.
\end{aligned}
$$

Consequents,

$$|v|_{l+1,T} \leq \frac{C_{2,k-l}}{h}|v|_{l,T}. \tag{4.2}$$

By analogy we obtain (4.1). □

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