UNIFORM QUADRATIC CONVERGENCE OF A MONOTONE WEIGHTED AVERAGE METHOD FOR SEMILINEAR SINGULARLY PERTURBED PARABOLIC PROBLEMS[*]

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Abstract

This paper deals with a monotone weighted average iterative method for solving semilinear singularly perturbed parabolic problems. Monotone sequences, based on the accelerated monotone iterative method, are constructed for a nonlinear difference scheme which approximates the semilinear parabolic problem. This monotone convergence leads to the existence-uniqueness theorem. An analysis of uniform convergence of the monotone weighted average iterative method to the solutions of the nonlinear difference scheme and continuous problem is given. Numerical experiments are presented.

Mathematics subject classification: 65M06, 65N06.
Key words: Semilinear parabolic problem, Singular perturbation, Weighted average scheme, Monotone iterative method, Uniform convergence.

1. Introduction

In this paper we give a numerical treatment for the semilinear singularly perturbed parabolic problem in the form

$$u_t - \mu^2(u_{xx} + u_{yy}) + f(x,y,t,u) = 0, \quad (x,y,t) \in Q = \omega \times (0,T], \tag{1.1a}$$

$$u(x,y,t) = g(x,y,t), \quad (x,y,t) \in \partial\omega \times (0,T], \tag{1.1b}$$

$$u(x,y,0) = \psi(x,y), \quad x \in \overline{\omega}, \tag{1.1c}$$

where $\omega = \{0 < x < 1\} \times \{0 < y < 1\}$, μ is a small positive parameter, $\partial\omega$ is the boundary of ω, the functions f, g and ψ are smooth in their respective domains, and f satisfies the constraint

$$f_u \geq 0, \quad (x,y,t,u) \in \overline{Q} \times (-\infty,\infty), \quad (f_u = \partial f/\partial u). \tag{1.2}$$

This assumption can always be obtained by a change of variables. Indeed, introduce

$$z(x,y,t) = \exp(-\lambda t)u(x,y,t),$$

where λ is a constant. Now, $z(x,y,t)$ satisfies (1.1) with

$$\varphi = \lambda z + \exp(-\lambda t)f(x,y,t,\exp(\lambda t)z),$$

instead of f, and we have $\varphi_z = \lambda + f_u$. Thus, if $\lambda \geq -\min f_u$, where minimum is taking over the domain from (1.2), we conclude $\varphi_z \geq 0$.

For $\mu \ll 1$, the problem is singularly perturbed and characterized by boundary layers (regions with rapid change of solutions) near boundary $\partial \omega$ (see [1] for details).

We shall employ the weighted average scheme for approximating the semilinear problem (1.1). This nonlinear ten-point difference scheme can be regarded as taking a weighted average of the explicit and implicit schemes. In order to practically compute the nonlinear weighted average scheme, one requires an efficient numerical method. A fruitful method for solving nonlinear difference schemes is the method of upper and lower solutions and its associated monotone iterations. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The above monotone iterative method is well known and has been widely used for continuous and discrete elliptic and parabolic boundary value problems. Most of publications on this topic involve monotone iterative schemes whose rate of convergence is of linear rate (cf. [2–7]) Some accelerated monotone iterative schemes for solving discrete elliptic boundary value problems are given in [8,9]. An advantage of this accelerated approach is that it leads to sequences which converge either quadratically or nearly quadratically. In [10], an accelerated monotone iterative method for solving discrete parabolic boundary value problems based on the implicit scheme is presented. In [11], a combination of the accelerated monotone iterative method from [10] with monotone Picard iterates is constructed. In [10, 11], the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were omitted.

In this paper, we extend the accelerated monotone iterative method from [10] to the case when on each time level a nonlinear difference scheme based on the weighted average of the explicit and implicit schemes is solved inexactly, and give an analysis of a convergence rate of this monotone iterative method. In [10], it is assumed that a pair of ordered upper and lower solutions is given on each time level, and this pair is used as initial iterates in the accelerated monotone iterative method. Our iterative method combines an explicit construction of initial upper and lower solutions on each time level and the modified accelerated monotone iterative method.

In [3], we investigate uniform convergence properties of the monotone weighted average iterative method applied to solving the semilinear problem (1.1). This monotone method possesses only linear convergence rate. In this paper, we investigate uniform convergence properties of the monotone weighted average iterative method based of the extended accelerated monotone iterative method from [10] . We show that the proposed monotone iterative method possesses quadratic convergence rate.

The structure of the paper as follows. In Section 2, we introduce the nonlinear weighted average scheme for the numerical solution of (1.1). The monotone weighted average iterative method is presented in Section 3. The explicit construction of initial upper and lower solutions is incorporated in the monotone weighted average iterative method. Section 4 deals with existence and uniqueness of the solution to the nonlinear difference scheme. In Section 5, we show that the monotone weighted average iterative method possesses uniform quadratic convergence rate. An analysis of convergence rates of the monotone weighted average iterative method, based of different stopping tests, is given in Section 6. Section 7 deals with uniform convergence of the monotone weighted average iterative method to the continuous problem (1.1). The final Section 8 presents results of numerical experiments where iteration counts are compared between the proposed monotone weighted average iterative method and monotone weighted

average iterative method from [3], whose convergence rate is linear.

2. The Weighted Average Scheme

On $\overline{Q}$ introduce a rectangular mesh $\overline{\omega}^h \times \overline{\omega}^\tau$, $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \left\{ x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i \right\}, \tag{2.1a}$$

$$\overline{\omega}^{hy} = \left\{ y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j \right\}, \tag{2.1b}$$

$$\overline{\omega}^\tau = \left\{ t_k,\ 0 \le k \le N_\tau;\ t_0 = 0,\ t_{N_\tau} = T;\ \tau_k = t_k - t_{k-1} \right\}. \tag{2.1c}$$

For solving (1.1), consider the weighted average method (or θ-method)

$$\tau_k^{-1} \left[U(p, t_k) - U(p, t_{k-1}) \right] + \theta \mathcal{L}^h U(p, t_k) + (1 - \theta) \mathcal{L}^h U(p, t_{k-1})$$
$$+ \theta f(p, t_k, U) + (1 - \theta) f(p, t_{k-1}, U) = 0, \quad (p, t_k) \in \omega^h \times \omega^\tau, \tag{2.2}$$

with the boundary and initial conditions

$$U(p, t_k) = g(p, t_k), \quad (p, t_k) \in \partial \omega^h \times \omega^\tau,$$
$$U(p, 0) = \psi(p), \qquad p \in \overline{\omega}^h,$$

where $\theta = \text{const}$ and $\partial \omega^h$ is the boundary of $\overline{\omega}^h$. When no confusion arises, we will write $f(p, t_k, U(p, t_k)) = f(p, t_k, U)$. $\mathcal{L}^h U$ is defined by

$$\mathcal{L}^h U = -\mu^2 \left(\mathcal{D}_x^2 U + \mathcal{D}_y^2 U \right),$$

where $\mathcal{D}_x^2 U$ and $\mathcal{D}_y^2 U$ are the central difference approximations to the second derivatives

$$\mathcal{D}_x^2 U_{ij}^k = (\hbar_{xi})^{-1} \left[\left(U_{i+1,j}^k - U_{ij}^k \right) (h_{xi})^{-1} - \left(U_{ij}^k - U_{i-1,j}^k \right) (h_{x,i-1})^{-1} \right],$$

$$\mathcal{D}_y^2 U_{ij}^k = (\hbar_{yj})^{-1} \left[\left(U_{i,j+1}^k - U_{ij}^k \right) (h_{yj})^{-1} - \left(U_{ij}^k - U_{i,j-1}^k \right) (h_{y,j-1})^{-1} \right],$$

where

$$\hbar_{xi} = 2^{-1} \left(h_{x,i-1} + h_{xi} \right), \quad \hbar_{yj} = 2^{-1} \left(h_{y,j-1} + h_{yj} \right), \quad U_{ij}^k \equiv U\left(x_i, y_j, t_k \right).$$

This 10-point difference scheme can be regarded as taking a weighted average of the explicit scheme ($\theta = 0$) and the fully implicit scheme ($\theta = 1$). We assume that we are using an average with nonnegative weights, so that $0 \le \theta \le 1$.

On each time level t_k, $k \ge 1$, introduce the linear difference problem

$$\left(\theta \mathcal{L}^h + \tau_k^{-1} + \theta c(p, t_k) \right) W(p, t_k) = \Phi(p, t_k), \quad p \in \omega^h, \tag{2.3a}$$

$$c(p, t_k) \ge 0, \quad W(p, t_k) = g(p, t_k), \qquad p \in \partial \omega^h. \tag{2.3b}$$

We now formulate the maximum principle and give an estimate to the solution of (2.3).

Lemma 2.1. *(i) If a mesh function $W(p, t_k)$ satisfies the conditions*

$$\left(\theta \mathcal{L}^h + \tau_k^{-1} + \theta c(p, t_k) \right) W(p, t_k) \ge 0 \ (\le 0), \quad p \in \omega^h,$$

$$W(p, t_k) \ge 0 \ (\le 0), \qquad p \in \partial \omega^h,$$

then $W(p, t_k) \geq 0$ (≤ 0) in $\overline{\omega}^h$.

(ii) The following estimates of the solutions to (2.3) hold true

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \leq \max\left\{ \|g(\cdot, t_k)\|_{\partial\omega^h}, \max_{p\in\omega^h}\left|\frac{\Phi(p, t_k)}{\theta c(p, t_k) + \tau_k^{-1}}\right| \right\}, \tag{2.4}$$

where

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} = \max_{p\in\overline{\omega}^h}|W(p, t_k)|, \quad \|g(\cdot, t_k)\|_{\partial\omega^h} = \max_{p\in\partial\omega^h}|g(p, t_k)|.$$

The proof of parts (i) and (ii) can be found in, respectively, [3] and [12].

3. The Monotone Iterative Method

We represent the difference equation from (2.2) in the equivalent form

$$\mathcal{G}(U(p, t_k), U(p, t_{k-1})) = 0, \quad (p, t) \in \omega^h \times \omega^\tau, \tag{3.1a}$$

$$\mathcal{G}(U(p, t_k), U(p, t_{k-1})) = \mathcal{G}_\theta(p, t_k, U) + \mathcal{G}_{1-\theta}(p, t_{k-1}, U), \tag{3.1b}$$

$$\mathcal{G}_\theta(p, t_k, U) = \left(\theta\mathcal{L}^h + \tau_k^{-1}\right)U(p, t_k) + \theta f(p, t_k, U), \tag{3.1c}$$

$$\mathcal{G}_{1-\theta}(p, t_{k-1}, U) = \left[(1-\theta)\mathcal{L}^h - \tau_k^{-1}\right]U(p, t_{k-1}) + (1-\theta)f(p, t_{k-1}, U). \tag{3.1d}$$

We say that on a time level t_k, $k \geq 1$, $V_1(p, t_k)$ is an upper solution with respect to a given function $V(p, t_{k-1})$, if it satisfies

$$\mathcal{G}(V_1(p, t_k), V(p, t_{k-1}) \geq 0, \quad p \in \omega^h,$$
$$V_1(p, t_k) \geq g(p, t_k), \qquad p \in \partial\omega^h.$$

Similarly, $V_{-1}(p, t_k)$ is called a lower solution with respect to a given function $V(p, t_{k-1})$, if it satisfies the reversed inequalities.

3.1. Statement of the monotone iterative method

We now construct an iterative method for solving (3.1) in the following way. On each time level t_k, $k \geq 1$, initial upper and lower solutions $V_\alpha^{(0)}(p, t_k)$ ($\alpha = 1$ and $\alpha = -1$ correspond to, respectively, the upper and lower cases) are calculated by solving the linear problems

$$\left(\theta\mathcal{L}^h + \tau_k^{-1}\right)W_\alpha^{(0)}(p, t_k) = \alpha|\mathcal{G}(S(p, t_k), V_1(p, t_{k-1}))|, \quad p \in \omega^h, \tag{3.2a}$$

$$W_\alpha^{(0)}(p, t_k) = 0, \quad p \in \partial\omega^h, \tag{3.2b}$$

$$V_\alpha^{(0)}(p, t_k) = S(p, t_k) + W_\alpha^{(0)}(p, t_k), \quad p \in \overline{\omega}^h, \quad \alpha = 1, -1, \tag{3.2c}$$

where $S(p, t_k)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $g(p, t_k)$ on $\partial\omega^h$. We calculate upper and lower sequences $\{V_\alpha^{(n)}(p, t_k)\}$, $\alpha = 1, -1$, for $n \geq 1$, by using the recurrence formulae

$$\mathcal{L}^{(n)}(p, t_k)Z_\alpha^{(n)}(p, t_k) = -\mathcal{G}(V_\alpha^{(n-1)}(p, t_k), V_1(p, t_{k-1})), \quad p \in \omega^h, \tag{3.3a}$$

$$\mathcal{L}^{(n)}(p, t_k) = \theta\mathcal{L}^h + \tau_k^{-1} + \theta c^{(n-1)}(p, t_k), \tag{3.3b}$$

$$Z_\alpha^{(n)}(p, t_k) = 0, \quad p \in \partial\omega^h, \tag{3.3c}$$

$$V_\alpha^{(n)}(p, t_k) = V_\alpha^{(n-1)}(p, t_k) + Z_\alpha^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \tag{3.3d}$$

$$V_\alpha(p, t_k) = V_\alpha^{(n_k)}(p, t_k), \quad p \in \overline{\omega}^h, \tag{3.3e}$$

$$V_\alpha(p, 0) = \psi(p), \quad p \in \overline{\omega}^h, \tag{3.3f}$$

where $\mathcal{G}(V_\alpha^{(n-1)}(p, t_k), V_1(p, t_{k-1}))$ is the residual of the difference scheme (3.1) on $V_\alpha^{(n-1)}$ for upper $\alpha = 1$ and lower $\alpha = -1$ sequences, respectively, and n_k is a number of iterative steps on time-level t_k. The mesh function $c^{(n-1)}(p, t_k)$ is given by

$$c^{(n-1)}(p, t_k) = \max_V \{f_u(p, t_k, V), \quad V_{-1}^{(n-1)}(p, t_k) \leq V \leq V_1^{(n-1)}(p, t_k)\}, \qquad (3.4)$$

where below in Theorem 3.1, we prove that

$$V_{-1}^{(n-1)}(p, t_k) \leq V_1^{(n-1)}(p, t_k), \quad p \in \overline{\omega}^h.$$

3.2. Monotone convergence of method (3.2)–(3.4)

If $U(p, t_k) \geq V(p, t_k)$, $p \in \overline{\omega}^h$, we define the following sector:

$$\langle V(t_k), U(t_k) \rangle = \{V(p, t_k) \leq W(p, t_k) \leq U(p, t_k), \quad p \in \overline{\omega}^h\}.$$

In the following theorem we prove the monotone property of the iterative method (3.2)–(3.4).

Theorem 3.1. *The sequences $\{V_1^{(n)}\}$, $\{V_{-1}^{(n)}\}$, generated by (3.2)–(3.4), are, respectively, upper and lower solutions and converge monotonically*

$$V_{-1}^{(n-1)}(p, t_k) \leq V_{-1}^{(n)}(p, t_k) \leq V_1^{(n)}(p, t_k) \leq V_1^{(n-1)}(p, t_k), \quad p \in \overline{\omega}^h, \qquad (3.5)$$

where $k \geq 1$ and $n \geq 1$.

Proof. We show that $V_1^{(0)}(p, t_k)$ defined by (3.2) is an upper solution. From the maximum principle in Lemma 2.1, it follows that $W_1^{(0)}(p, t_k) \geq 0$ on $\overline{\omega}^h$. Now, using the difference equation (3.2) for $W_1^{(0)}(p, t_k)$ and the mean-value theorem, we have

$$\mathcal{G}(S(p, t_k) + W_1^{(0)}(p, t_k), V_1(p, t_{k-1}))$$
$$= \mathcal{G}(S(p, t_k), V(p, t_{k-1})) + |\mathcal{G}(S(p, t_k), V(p, t_{k-1}))| + \theta f_u(p, t_k, E^{(0)})W_1^{(0)}(p, t_k),$$

where $E^{(0)}(t_k) \in \langle S(t_k), S(t_k) + W_1^{(0)}(t_k) \rangle$. Since f_u, $W_1^{(0)}$ and θ are nonnegative, we conclude that $V_1^{(0)}(p, t_k) = S(p, t_k) + W_1^{(0)}(p, t_k)$ is an upper solution. Similarly, we can prove that $V_{-1}^{(0)}(p, t_k) = S(p, t_k) + W_{-1}^{(0)}(p, t_k)$ is a lower solution, where $W_{-1}^{(0)}$ is nonpositive. Thus, $V_1^{(0)}(p, t_k)$ and $V_{-1}^{(0)}(p, t_k)$ are, respectively, upper and lower solutions of (3.1) and satisfy (3.5).

Since $V_1^{(0)}$ is an upper solution, then from (3.3) we conclude that

$$\mathcal{L}^{(1)}(p, t_k)Z_1^{(1)}(p, t_k) \leq 0, \quad p \in \omega^h, \quad Z_1^{(1)}(p, t_k) = 0, \quad p \in \partial\omega^h.$$

From Lemma 2.1, it follows that

$$Z_1^{(1)}(p, t_k) \leq 0, \quad p \in \overline{\omega}^h. \qquad (3.6)$$

Similarly, for a lower solution $V_{-1}^{(0)}$, we conclude that

$$Z_{-1}^{(1)}(p, t_k) \geq 0, \quad p \in \overline{\omega}^h. \qquad (3.7)$$

We now prove that

$$V_{-1}^{(1)}(p, t_k) \leq V_1^{(1)}(p, t_k), \quad p \in \overline{\omega}^h. \qquad (3.8)$$

Letting $W^{(n)} = V_1^{(n)} - V_{-1}^{(n)}$, $n \geq 0$, from (3.3), by the mean-value theorem, we have

$$\mathcal{L}^{(1)}(p,t_k)W^{(1)}(p,t_k) = \theta \left[c^{(0)}(p,t_k) - f_u(p,t_k,Q^{(0)}) \right] W^{(0)}(p,t_k), \quad p \in \omega^h,$$

$$W^{(1)}(p,t_k) = 0, \quad p \in \partial\omega^h,$$

where $Q^{(0)}(t_k) \in \langle V_{-1}^{(0)}(t_k), V_1^{(0)}(t_k) \rangle$. From here, taking into account $V_1^{(0)}(p,t_k) \geq V_{-1}^{(0)}(p,t_k)$ and (3.4) with $n = 1$, we conclude that the right hand side in the difference equation is nonnegative. The positivity property in Lemma 2.1 implies $W^{(1)}(p,t_k) \geq 0$, and this leads to (3.8).

We now prove that $V_1^{(1)}(p,t_k)$ and $V_{-1}^{(1)}(p,t_k)$ are upper and lower solutions, respectively. Using the mean-value theorem, from (3.3) we obtain

$$\mathcal{G}(V_1^{(1)}(p,t_k),V_1(p,t_{k-1})) = -\theta \left[c^{(0)}(p,t_k) - f_u(p,t_k,E^{(1)}) \right] Z_1^{(1)}(p,t_k), \qquad (3.9)$$

where $E^{(1)}(t_k) \in \langle V_1^{(1)}(t_k), V_1^{(0)}(t_k) \rangle$. From here, (3.4), (3.6)–(3.8), it follows that

$$c^{(0)}(p,t_k) \geq f_u(p,t_k,E^{(1)}), \quad p \in \overline{\omega}^h.$$

From here and (3.6), we conclude that

$$\mathcal{G}(V_1^{(1)}(p,t_k),V_1(p,t_{k-1})) \geq 0, \quad p \in \omega^h, \quad V_1^{(1)}(p,t_k) = g(p,t_k), \quad p \in \partial\omega^h.$$

Thus, $V_1^{(1)}(p,t_k)$ is an upper solution. Similarly, we can prove that $V_{-1}^{(1)}(p,t_k)$ is a lower solution, that is,

$$\mathcal{G}(V_{-1}^{(1)}(p,t_k),V_1(p,t_{k-1})) \leq 0, \quad p \in \omega^h, \quad V_{-1}^{(1)}(p,t_k) = g(p,t_k), \quad p \in \partial\omega^h.$$

By induction on n, we can prove that $\{V_1^{(n)}(p,t_k)\}$ is a monotonically decreasing sequence of upper solutions and $\{V_{-1}^{(n)}(p,t_k)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (3.5). Thus, we prove the theorem. $\square$

4. Existence and Uniqueness of a Solution to the Nonlinear Difference Scheme

Applying Theorem 3.1, we investigate existence and uniqueness of a solution to the nonlinear difference scheme (3.1).

Theorem 4.1. *The nonlinear difference scheme* (3.1) *has a unique solution.*

Proof. From (3.5), it follows that $\lim V_1^{(n)}(p,t_1) = U_1(p,t_1)$, $p \in \overline{\omega}^h$ as $n \to \infty$ exists, and

$$U_1(p,t_1) \leq V_1^{(n)}(p,t_1), \quad \lim_{n\to\infty} Z_1^{(n)}(p,t_1) = 0, \quad p \in \overline{\omega}^h. \qquad (4.1)$$

Similar to (3.9), we can prove that

$$\mathcal{G}(V_1^{(n)}(p,t_1),\psi(p)) = -\theta \left[c^{(0)}(p,t_1) - f_u(p,t_1,E^{(n)}) \right] Z_1^{(n)}(p,t_1), \quad n \geq 1,$$

where $E^{(n)}(t_1) \in \langle V_1^{(n)}(t_1), V_1^{(n-1)}(t_1) \rangle$, and ψ is the initial function from (1.1). From here and (4.1), it follows that $U_1(p, t_1)$ solves (3.1) at t_1. At time level t_2, we consider (3.2)–(3.4) with $V_1(p, t_1) = U_1(p, t_1)$. Using a similar argument, we can prove that the following limit

$$\lim_{n \to \infty} V_1^{(n)}(p, t_2) = U_1(p, t_2), \quad p \in \overline{\omega}^h,$$

exists and solves (3.1) at t_2, where according to Theorem 3.1, $\{V_1^{(n)}(p, t_2)\}$ is a sequence of upper solutions with respect to $U_1(p, t_1)$.

By induction on k, $k \geq 1$, we can prove that

$$U_1(p, t_k) = \lim_{n \to \infty} V_1^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \quad k \geq 1,$$

is a solution of the nonlinear difference scheme (3.1). Similarly, we can prove that the mesh function $U_{-1}(p, t_k)$ defined by

$$U_{-1}(p, t_k) = \lim_{n \to \infty} V_{-1}^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \quad k \geq 1,$$

is a solution of the nonlinear difference scheme (3.1). We now show that

$$U_1(p, t_k) = U_{-1}(p, t_k), \quad p \in \overline{\omega}^h, \quad k \geq 1, \tag{4.2}$$

where $U_1(p, t_k)$ and $U_{-1}(p, t_k)$ are solutions to the difference scheme (3.1), which are defined above. From the definition of $U_1(p, t_1)$ and $U_{-1}(p, t_1)$, we conclude that

$$\mathcal{G}(U_1(p, t_1), \psi(p)) = 0, \quad \mathcal{G}(U_{-1}(p, t_1), \psi(p)) = 0, \quad p \in \omega^h.$$

Letting $W(p, t_1) = U_1(p, t_1) - U_{-1}(p, t_1)$, from here and (3.1), we have

$$(\theta \mathcal{L}^h + \tau_1^{-1})W(p, t_1) + \theta[f(p, t_1, U_1) - f(p, t_1, U_{-1})] = 0, \quad p \in \omega^h,$$
$$W(p, t_1) = 0, \quad p \in \partial \omega^h.$$

Using the mean-value theorem, we obtain

$$(\theta \mathcal{L}^h + \tau_1^{-1} + \theta f_u(p, t_1, Q))W(p, t_1) = 0, \quad p \in \omega^h,$$
$$W(p, t_1) = 0, \quad p \in \partial \omega^h,$$

where $Q(t_1) \in \langle U_{-1}(t_1), U_1(t_1) \rangle$. Since f_u and θ are nonnegative, by Lemma 2.1, we conclude that $W(p, t_1) = 0$, $p \in \overline{\omega}^h$. By induction on k, $k \geq 1$, we can prove (4.2). Thus, the theorem holds true.   □

5. Uniform Quadratic Convergence

Introduce the notation

$$r_k = \max_{p \in \overline{\omega}^h} \left(\max_V \left\{ |f_{uu}(p, t_k, V)|, V_{-1}^{(0)}(p, t_k) \leq V \leq V_1^{(0)}(p, t_k) \right\} \right). \tag{5.1}$$

The following theorem gives the quadratic convergence of the monotone iterative method (3.2)–(3.4).

Theorem 5.1. *On each time level, for the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by (3.2)–(3.4), the following estimate holds:*

$$\|W^{(n+1)}(\cdot, t_k)\|_{\overline{\omega}^h} \leq \tau_k(\theta r_k)\|W^{(n)}(\cdot, t_k)\|_{\overline{\omega}^h}^2, \tag{5.2}$$

where $W^{(n)}(p, t_k) = V_1^{(n)}(p, t_k) - V_{-1}^{(n)}(p, t_k)$.

Proof. From (3.3), we have

$$\mathcal{L}^{(n+1)}(p, t_k)W^{(n+1)}(p, t_k) = H^{(n)}(p, t_k), \quad p \in \omega^h, \tag{5.3a}$$

$$H^{(n)}(p, t_k) = \theta c^{(n)}(p, t_k)W^{(n)}(p, t_k) - \theta[f(p, t_k, V_1^{(n)}) - f(p, t_k, V_{-1}^{(n)})], \tag{5.3b}$$

$$W^{(n+1)}(p, t_k) = 0, \quad p \in \partial\omega^h. \tag{5.3c}$$

By the mean-value theorem,

$$f(p, t_k, V_1^{(n)}) - f(p, t_k, V_{-1}^{(n)}) = f_u(p, t_k, Q^{(n)})W^{(n)}(p, t_k),$$

where $Q^{(n)}(t_k) \in \langle V_{-1}^{(n)}(t_k), V_1^{(n)}(t_k)\rangle$. From (3.4), it follows that

$$c^{(n)}(p, t_k) = f_u(p, t_k, G^{(n)}), \quad G^{(n)}(t_k) \in \langle V_{-1}^{(n)}(t_k), V_1^{(n)}(t_k)\rangle.$$

Thus, we represent the right hand side $H^{(n)}$ from (5.3) in the form

$$\theta[f_u(p, t_k, G^{(n)}) - f_u(p, t_k, Q^{(n)})]W^{(n)}(p, t_k).$$

Applying again the mean-value theorem, we get

$$f_u(p, t_k, G^{(n)}) - f_u(p, t_k, Q^{(n)}) = f_{uu}(p, t_k, R^{(n)})(G^{(n)}(p, t_k) - Q^{(n)}(p, t_k)),$$

where $R^{(n)}$ lies between $G^{(n)}$ and $Q^{(n)}$. Taking into account that

$$|G^{(n)}(p, t_k) - Q^{(n)}(p, t_k)| \leq V_1^{(n)}(p, t_k) - V_{-1}^{(n)}(p, t_k),$$

in the notation (5.1), we estimate $H^{(n)}$ from (5.3) as follows

$$\|H^{(n)}(\cdot, t_k)\|_{\omega^h} \leq \theta r_k\|W^{(n)}(\cdot, t_k)\|_{\overline{\omega}^h}^2.$$

From here and $f_u \geq 0$, using (2.4), we prove the estimate (5.2). $\qquad\square$

Remark 5.1. If on each time level t_k, $k \geq 1$, the nonlinear function f satisfies the constraint

$$\min_{p \in \overline{\omega}^h} \left(\min_V \left\{ f_{uu}(p, t_k, V), V_{-1}^{(0)}(p, t_k) \leq V \leq V_1^{(0)}(p, t_k)\right\}\right) \geq 0, \tag{5.4}$$

then for the upper sequence $\{V_1^{(n)}\}$ in Theorem 5.1, we have the estimate

$$\|V_1^{(n+1)}(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h} \leq \tau_k(\theta r_k)\|V_1^{(n)}(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h}^2,$$

where $U^*(p, t_k)$ is the exact solution of the nonlinear difference scheme (3.1). From the assumption (5.4) on f_{uu} and (3.4), it follows that

$$c^{(n)}(p, t_k) = f_u(p, t_k, V_1^{(n)}). \tag{5.5}$$

If we take into account that $W^{(n)} = V_1^{(n)} - U^*$ and $U^* \leq E^{(n)} \leq V_1^{(n)}$, the proof of the estimate repeats the proof of Theorem 5.1.

If on each time level t_k, $k \geq 1$, the nonlinear function f satisfies the constrain

$$\max_{p \in \overline{\omega}^h} \left(\max_V \left\{ f_{uu}(p, t_k, V), V_{-1}^{(0)}(p, t_k) \leq V \leq V_1^{(0)}(p, t_k) \right\} \right) \leq 0,$$

then for the lower sequence $\{V_{-1}^{(n)}\}$ in Theorem 5.1, we have the estimate

$$\|V_{-1}^{(n+1)}(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h} \leq \tau_k(\theta r_k) \|V_{-1}^{(n)}(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h}^2.$$

From the assumption (5.4) on f_{uu} and (3.4), it follows that

$$c^{(n)}(p, t_k) = f_u(p, t_k, V_{-1}^{(n)}).$$

If we take into account that $W^{(n)} = V_{-1}^{(n)} - U^*$ and $V_{-1}^{(n)} \leq E^{(n)} \leq U^*$, the proof of the estimate repeats the proof of Theorem 5.1.

Without loss of generality, we assume that the boundary condition is zero, that is, $g(x, y, t) = 0$ in (1.1). This assumption can always be satisfied by a change of variables. On each time level, from (3.2), we have

$$\left(\theta \mathcal{L}^h + \tau_k^{-1} \right) V_\alpha^{(0)}(p, t_k) = \alpha |\mathcal{G}(0, V_1(p, t_{k-1}))|, \quad p \in \omega^h, \tag{5.6a}$$

$$V_\alpha^{(0)}(p, t_k) = 0, \quad p \in \partial \omega^h, \quad \alpha = 1, -1, \tag{5.6b}$$

where $S(p, t_k) = 0$.

Lemma 5.1. *On each time level $k \geq 1$, upper and lower solutions $V_\alpha^{(n)}(p, t_k)$, $\alpha = 1, -1$, $n \geq 0$, obtained by (5.6), (3.3) and (3.4), are bounded independently of μ, N_x, N_y and τ_k.*

Proof. We prove this result by induction. From (3.1) and (5.6), it follows that

$$\left(\theta \mathcal{L}^h + \tau_1^{-1} \right) V_\alpha^{(0)}(p, t_1) = \alpha |\theta f(p, t_1, 0) + \mathcal{G}_{1-\theta}(p, 0, \psi)|, \quad p \in \omega^h,$$

$$V_\alpha^{(0)}(p, t_1) = 0, \quad p \in \partial \omega^h,$$

$$\mathcal{G}_{1-\theta}(p, 0, \psi) = \left[(1 - \theta)\mathcal{L}^h - \tau_1^{-1} \right] \psi(p) + (1 - \theta)f(p, 0, \psi).$$

From here by (2.4), we have

$$\|V_\alpha^{(0)}(\cdot, t_1)\|_{\overline{\omega}^h} \leq \tau_1 \left[\|\theta f(\cdot, t_1, 0)\|_{\overline{\omega}^h} + \|\mathcal{G}_{1-\theta}(\cdot, t_1, \psi)\|_{\omega^h} \right]. \tag{5.7}$$

In [13], we proved the following estimates:

$$\|\mu^2 \mathcal{D}_x^2(\psi)\|_{\omega^h} \leq \|\mu^2 \psi_{xx}\|_\omega, \quad \|\mu^2 \mathcal{D}_y^2(\psi)\|_{\omega^h} \leq \|\mu^2 \psi_{yy}\|_\omega.$$

It means that $\|\mathcal{L}^h \psi\|_{\omega^h}$ is μ-uniformly bounded. Thus,

$$\tau_1 \|\mathcal{G}_{1-\theta}(\cdot, 0, \psi)\|_{\omega^h} \leq C, \tag{5.8}$$

where here and throughout a positive generic constant C is independent of μ, N_x, N_y and τ_k. From here and (5.7), it follows that

$$\|V_\alpha^{(0)}(\cdot, t_1)\|_{\overline{\omega}^h} \leq C, \quad \alpha = 1, -1. \tag{5.9}$$

From (3.3), for $\alpha = 1, -1$, we have

$$
\begin{aligned}
&\mathcal{L}^{(1)}(p, t_1) V_\alpha^{(1)}(p, t_1) \\
&= \theta c^{(0)}(p, t_1) V_\alpha^{(0)}(p, t_1) - \theta f(p, t_1, V_\alpha^{(0)}) + \mathcal{G}_{1-\theta}(p, 0, \psi), \quad p \in \omega^h.
\end{aligned} \tag{5.10}
$$

By (2.4),

$$
\begin{aligned}
&\|V_\alpha^{(1)}(\cdot, t_1)\|_{\overline{\omega}^h} \\
&\leq \tau_1 [\theta \| c^{(0)}(\cdot, t_1)\|_{\overline{\omega}^h} \|V_\alpha^{(0)}(\cdot, t_1)\|_{\overline{\omega}^h} + \theta \|f(\cdot, t_1, V_\alpha^{(0)})\|_{\overline{\omega}^h} + \|\mathcal{G}_{1-\theta}(\cdot, 0, \psi)\|_{\omega^h}].
\end{aligned}
$$

From here, (3.4), (5.8)–(5.10), we conclude that

$$
\|V_\alpha^{(1)}(\cdot, t_1)\|_{\overline{\omega}^h} \leq C, \quad \tau_1 \|\mathcal{L}^h V_\alpha^{(1)}(\cdot, t_1\|_{\omega^h} \leq C, \quad \alpha = 1, -1.
$$

Now, by induction on n, we prove that $V_\alpha(p, t_1) = V_\alpha^{(n_1)}(p, t_1)$ satisfies the estimates

$$
\|V_\alpha(\cdot, t_1)\|_{\overline{\omega}^h} \leq C, \quad \tau_1 \|\mathcal{L}^h V_\alpha(\cdot, t_1)\|_{\omega^h} \leq C, \quad \alpha = 1, -1.
$$

Using these estimates, we conclude that

$$
\tau_2 \mathcal{G}_{1-\theta}(p, t_1, V_1) = \tau_2 \left[(1 - \theta) \mathcal{L}^h - \tau_2^{-1} \right] V_1(p, t_1) + \tau_2 (1 - \theta) f(p, t_1, V_1).
$$

is bounded independently of μ, N_x, N_y and τ_2 (compare with (5.8)). Now, by induction on k, we prove the lemma. □

We now prove that on each time level the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, obtained by (5.6), (3.3) and (3.4) converge uniformly in the perturbation parameter μ. Introduce the notation

$$
q_n(t_k) = \|V_1^{(n)}(\cdot, t_k) - V_{-1}^{(n)}(\cdot, t_k)\|_{\overline{\omega}^h}. \tag{5.11}
$$

Theorem 5.2. *On each time level k, $k \geq 1$, the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by (5.6), (3.3) and (3.4) converge μ-uniformly. There exists n_k, such that $\tau_k r_k q_{n_k} < 1$, and the following estimate holds:*

$$
q_n(t_k) \leq \frac{1}{\theta \tau_k r_k} \left[\theta \tau_k r_k q_{n_k}(t_k) \right]^{2^{n-n_k}}, \quad n \geq n_k, \tag{5.12}
$$

where r_k is defined in (5.1).

Proof. From (3.5) and Lemma 5.1, we conclude that the upper and lower solutions converge μ-uniformly. Let $\kappa_n(t_k) = \theta \tau_k r_k q_n(t_k)$. Multiplying (5.2) by $\theta \tau_k r_k$, we have

$$
\kappa_{n+1}(t_k) \leq [\kappa_n(t_k)]^2, \quad n \geq 0.
$$

Since the sequences $\{V_\alpha^{(n)}(p, t_k)\}$, $\alpha = 1, -1$, converge to the solution $U^*(p, t_k)$ of the nonlinear scheme (3.1), then for some n_k the inequality $\kappa_{n_k} < 1$ holds. By induction, we show that

$$
\kappa_n(t_k) \leq [\kappa_{n_k}(t_k)]^{2^{n-n_k}}, \quad n \geq n_k. \tag{5.13}
$$

It is true for $n = n_k$. Assuming that it holds true for $n = l$, we have

$$
\kappa_{l+1}(t_k) \leq [\kappa_l(t_k)]^2 \leq \left([\kappa_{n_k}(t_k)]^{2^{l-n_k}} \right)^2 = [\kappa_{n_k}(t_k)]^{2^{l+1-n_k}},
$$

and prove (5.13). From (5.13) and $\kappa_n(t_k) = \theta \tau_k r_k q_n(t_k)$, we conclude (5.12). □

6. Convergence Analysis of the Monotone Iterative Method

In Theorems 3.1, 5.1 and 5.2, we have investigated convergence properties of method (3.2)–(3.4) on each time-level t_k, $k \geq 1$. In this section, we investigate convergence properties of the monotone iterative method (3.2)–(3.4) on the whole interval of integration $[0, T]$. In (3.3), we assume that on each time-level t_k, $k \geq 1$, $V(p, t_k) = V_1(p, t_k)$, where $V_1(p, t_k) = V_1^{(n_k)}(p, t_k)$. Thus, on the whole interval of integration, we now estimate $\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h}$, where U^* is the exact solution to the nonlinear difference scheme (3.1). In the final remark to this section, we formulate convergence results for the case when on each time-level t_k, $k \geq 1$, $V(p, t_k) = V_{-1}(p, t_k)$, where $V_{-1}(p, t_k) = V_{-1}^{(n_k)}(p, t_k)$.

6.1. Stopping criterion based on residual

We now choose the stopping criterion of the iterative method (3.2)–(3.4) in the form

$$\|\mathcal{G}(V_1^{(n)}(\cdot, t_k), V_1(\cdot, t_{k-1}))\|_{\omega^h} \leq \delta, \tag{6.1}$$

where δ is a prescribed accuracy, and set up $V_1(p, t_k) = V_1^{(n_k)}(p, t_k)$, $p \in \overline{\omega}^h$, such that n_k is minimal subject to (6.1).

Introduce the notation

$$v_i = v_{x,i-1} + v_{xi}, \qquad v_{x,i-1} = (\hbar_{xi})^{-1} (h_{x,i-1})^{-1}, \qquad v_{xi} = (\hbar_{xi})^{-1} (h_{xi})^{-1}, \tag{6.2a}$$

$$w_j = w_{y,j-1} + w_{yj}, \quad w_{y,j-1} = (\hbar_{yj})^{-1} (h_{y,j-1})^{-1}, \quad w_{yj} = (\hbar_{yj})^{-1} (h_{yj})^{-1}, \tag{6.2b}$$

$$\overline{v} = \max_{1 \leq i \leq N_x - 1} v_i, \qquad \overline{w} = \max_{1 \leq j \leq N_y - 1} w_j. \tag{6.2c}$$

We suppose that the time mesh spacing τ_k, $k \geq 1$ satisfies the constraint

$$(1 - \theta)\tau_k \leq \frac{1}{\mu^2 (\overline{v} + \overline{w}) + c_{k-1}}, \tag{6.3a}$$

$$c_{k-1} = \max_{p \in \overline{\omega}^h} \left(\max_V \left\{ f_u(p, t_{k-1}, V), V_{-1}^{(0)}(p, t_{k-1}) \leq V \leq V_1^{(0)}(p, t_{k-1}) \right\} \right), \quad k \geq 1. \tag{6.3b}$$

The condition (6.3), known as the CFL condition, guarantees the discrete maximum principle on the computational domain $\overline{\omega}^h \times \overline{\omega}^\tau$ (see in [3] for details). This imposes no time step restriction on the implicit scheme, for which $\theta = 1$. A more interesting question is the stability of the Crank-Nicolson scheme [14], for which $\theta = 0.5$. For a linear problem (1.1) with constant coefficients, Fourier analysis places no stability restriction on the Crank-Nicolson scheme, in contrast to condition (6.3). One can see that the CFL condition (6.3) is sharp by considering the one-dimensional linear problem $-\mu^2 u_{xx} + u_t = 0$ with initial data $u^0(x) = \{2x, 0 \leq x \leq 0.5; 2(1 - x), 0.5 \leq x \leq 1\}$, boundary conditions $g(0, t) = g(1, t) = 0$ and $N_x = 2$ mesh intervals [1].

Thus the maximum principle analysis can be viewed as an alternative means of obtaining stability conditions. It has the advantage over Fourier analysis that it is easily extended to problems with variable coefficients and to nonlinear problems. We mention that, in general, the maximum principle analysis gives only sufficient conditions for stability of difference schemes.

We prove the following convergence result for the monotone iterative method (3.2)–(3.4), (6.1).

Theorem 6.1. *Let* (6.3) *hold true. For the sequence* $\{V_1^{(n)}\}$, *generated by* (3.2)–(3.4), (6.1), *the following estimate holds:*

$$\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h} \le T\delta, \tag{6.4}$$

where $U^*(p, t_k)$ *is the unique solution to* (3.1). *Furthermore, on each time level the sequences converge monotonically* (3.5).

Proof. The monotone convergence of the sequence $\{V_1^{(n)}(p, t_k)\}$ follows from Theorem 3.1. The existence and uniqueness of the solution to (3.1) have been proved in Theorem 4.1.

From (3.1), we have

$$\mathcal{G}_\theta(p, t_k, V_1) + \mathcal{G}_{1-\theta}(p, t_{k-1}, V_1) = \mathcal{G}(V_1(p, t_k), V_1(p, t_{k-1})),$$
$$\mathcal{G}_\theta(p, t_k, U^*) + \mathcal{G}_{1-\theta}(p, t_{k-1}, U^*) = 0.$$

From here and using the mean-value theorem, we get the difference problem for $W(p, t_k) = V_1(p, t_k) - U^*(p, t_k)$ in the form

$$\left(\theta \mathcal{L}_f^h(p, t_k) + \tau_k^{-1}\right) W(p, t_k)$$
$$= -\left[(1-\theta)\mathcal{L}_f^h(p, t_{k-1}) - \tau_k^{-1}\right] W(p, t_{k-1}) + \mathcal{G}(V_1(p, t_k), V_1(p, t_{k-1})), \quad p \in \omega^h,$$
$$W(p, t_k) = 0, \quad p \in \partial\omega^h, \quad \mathcal{L}_f^h(p, t_k) = \mathcal{L}^h + f_u(p, t_k, E),$$

where $E(t_k) \in \langle U^*(t_k), V_1(t_k)\rangle$. In the notation of (6.2), we can write the difference equation in the form

$$(1 + \theta\tau_k\rho_{ij}^k)W_{ij}^k$$
$$= \theta\tau_k\left[\mathcal{M}_x(W_{ij}^k) + \mathcal{M}_y(W_{ij}^k)\right] + (1-\theta)\tau_k\left[\mathcal{M}_x(W_{ij}^{k-1}) + \mathcal{M}_y(W_{ij}^{k-1})\right]$$
$$+ \left[1 - (1-\theta)\tau_k\rho_{ij}^{k-1}\right] W_{ij}^{k-1} + \tau_k\mathcal{G}(V_1(x_i, y_j, t_k), V_1(x_i, y_j, t_{k-1})), \tag{6.5a}$$
$$\rho_{ij}^k = \mu^2(v_i + w_j) + f_{u,ij}^k, \quad \mathcal{M}_x(W_{ij}^k) = \mu^2\left(v_{xi}W_{i+1,j}^k + v_{x,i-1}W_{i-1,j}^k\right), \tag{6.5b}$$
$$\mathcal{M}_y(W_{ij}^k) = \mu^2\left(w_{yj}W_{i,j+1}^k + w_{y,j-1}W_{i,j-1}^k\right). \tag{6.5c}$$

Under the assumptions of the theorem, all the coefficients on the right hand side are nonnegative. From here and taking into account that according to Theorem 3.1 the stopping criterion (6.1) can always be satisfied, we have

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \le \delta\tau_k + \|W(\cdot, t_{k-1})\|_{\overline{\omega}^h}.$$

Taking into account that $\|W(\cdot, t_0)\|_{\overline{\omega}^h} = 0$, by induction on k, we conclude that

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \le \delta\sum_{l=1}^{k}\tau_l \le T\delta, \quad k \ge 1.$$

Thus, we prove the theorem. $\qquad\square$

6.2. Stopping criterion with fixed number of iterates on each time level

For the iterative method (3.2)–(3.4), we now choose the stopping criterion with the fixed number of iterative steps n_* on each time level (that is, n_* is independent of k) and assume that time step τ_k satisfies the inequality

$$\tau_k < \frac{1}{\theta c_k}, \quad k \ge 1, \tag{6.6}$$

where c_k is defined in (6.3).

Lemma 6.1. *Let* (6.6) *hold. Then for the sequences* $\{Z_1^{(n)}\}$, *generated by* (3.2)–(3.4)*, the following estimate holds:*

$$\|Z_1^{(n)}(\cdot, t_k)\|_{\overline{\omega}^h} \leq q_k^{n-1} \|Z_1^{(1)}(\cdot, t_k)\|_{\overline{\omega}^h}, \quad q_k = (\theta c_k)\tau_k < 1. \tag{6.7}$$

Proof. Similar to (3.9), using the mean-value theorem, from (3.3) we obtain

$$\mathcal{G}(V_1^{(n)}(p, t_k), V_1(p, t_{k-1})) = -\theta[c^{(n-1)}(p, t_k) - f_u(p, t_k, E^{(n)})]Z_1^{(n)}(p, t_k), \tag{6.8}$$

where $E^{(n)}(t_k) \in \langle V_1^{(n)}(t_k), V_1^{(n-1)}(t_k)\rangle$. Using (2.4), from (3.3) we have

$$\|Z_1^{(n)}(\cdot, t_k)\|_{\overline{\omega}^h} \leq \tau_k \|\mathcal{G}(V_1^{(n-1)}(\cdot, t_k), V_1(\cdot, t_{k-1}))\|_{\omega^h}.$$

From (3.4), $f_u \geq 0$ and (6.8), we conclude that

$$\|\mathcal{G}(V_1^{(n-1)}(\cdot, t_k), V_1(\cdot, t_{k-1}))\|_{\omega^h} \leq \theta c_k \|Z_1^{(n-1)}(\cdot, t_k)\|_{\overline{\omega}^h}.$$

Thus,

$$\|Z_1^{(n)}(\cdot, t_k)\|_{\overline{\omega}^h} \leq (\theta c_k)\tau_k \|Z_1^{(n-1)}(\cdot, t_k)\|_{\overline{\omega}^h},$$

and, by induction on n, we prove (6.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.2. *Let the assumptions* (6.3) *and* (6.6) *be satisfied. Then for the sequence* $\{V_1^{(n)}\}$ *generated by* (5.6), (3.3) *and* (3.4) *and* $n_* = $ *fixed, the following estimate holds:*

$$\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h} \leq Cq^{n_*-1}, \quad q = \max_{k \geq 1} q_k < 1, \tag{6.9}$$

where $U^*(p, t_k)$ *is the unique solution to* (3.1)*,* q_k *is defined in* (6.7)*, constant* C *is independent of* μ, N_x, N_y *and* τ_k*, and the number of iterative steps on each time level* $n_* \geq 2$*. Furthermore, on each time level the sequences converge monotonically* (3.5)*.*

Proof. From (6.5) and (6.8), we get the estimate for $W(p, t_k) = V_1(p, t_k) - U^*(p, t_k)$, $V_1(p, t_k) = V_1^{(n_*)}(p, t_k)$,

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \leq (\theta c_k)\tau_k \|Z_1^{(n_*)}(\cdot, t_k)\|_{\overline{\omega}^h} + \|W(\cdot, t_{k-1})\|_{\overline{\omega}^h}, \tag{6.10}$$

where c_k is defined in (6.3). From here and taking into account that $W(p, t_0) = 0$, we have

$$\|W(\cdot, t_1)\|_{\overline{\omega}^h} \leq (\theta c_1)\tau_1 \|Z_1^{(n_*)}(\cdot, t_1)\|_{\overline{\omega}^h}.$$

From here and (6.7), we obtain the estimate

$$\|W(\cdot, t_1)\|_{\overline{\omega}^h} \leq q_1^{n_*} \|Z_1^{(1)}(\cdot, t_1)\|_{\overline{\omega}^h}.$$

By Lemma 5.1, $\|Z_1^{(1)}(\cdot, t_1)\|_{\overline{\omega}^h}$ and $|c_1|$ are bounded independently of μ, N_x, N_y and τ_1. Thus, we obtain the estimate

$$\|W(\cdot, t_1)\|_{\overline{\omega}^h} \leq B_1\tau_1 q_1^{n_*-1}, \tag{6.11}$$

where here and throughout constants B_k, $k \geq 1$, are independent of μ, N_x, N_y and τ_k.

From (6.7) and (6.10), we obtain the estimate

$$\|W(\cdot, t_2)\|_{\overline{\omega}^h} \leq q_2^{n_*} \|Z_1^{(1)}(\cdot, t_2)\|_{\overline{\omega}^h} + \|W(\cdot, t_1)\|_{\overline{\omega}^h}. \tag{6.12}$$

By Lemma 5.1, $\|Z_1^{(2)}(\cdot, t_2)\|_{\overline{\omega}^h}$ and $|c_2|$ are bounded independently of μ, N_x, N_y and τ_1. Thus, from (6.11) and (6.12), we obtain the estimate

$$\|W(\cdot, t_2)\|_{\overline{\omega}^h} \leq B_1 \tau_1 q_1^{n_*-1} + B_2 \tau_2 q_2^{n_*-1}.$$

By induction on k, we can prove

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} = \sum_{s=1}^{k} B_s \tau_s q_s^{n_*-1}.$$

Denoting

$$B = \max_{k \geq 1} B_k,$$

and taking into account that $\sum_{s=1}^{k} \tau_s \leq T$, we prove (6.9) with $C = BT$.   □

Remark 6.1. We now assume that in (3.3), on each time-level t_k, $k \geq 1$, $V(p, t_k) = V_{-1}(p, t_k)$, where $V_{-1}(p, t_k) = V_{-1}^{(n_k)}(p, t_k)$. In this case, Theorems 6.1 and 6.2 hold true for

$$\max_{t_k \in \overline{\omega}^\tau} \|V_{-1}(\cdot, t_k) - U^*(\cdot, t_k)\|_{\overline{\omega}^h},$$

where $U^*(p, t_k)$ is the exact solution to the nonlinear difference scheme (3.1).

7. Uniform Convergence to the Solution of Problem (1.1)

We employ a layer-adapted mesh of a piecewise uniform type [15]. The piecewise uniform meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are defined in the manner of [15] and are referred to as Shishkin meshes. The boundary layer thicknesses ς_x and ς_y are chosen as

$$\varsigma_x = \min\{0.25, m_1 \mu \ln N_x\}, \quad \varsigma_y = \min\{0.25, m_2 \mu \ln N_y\}, \tag{7.1}$$

where m_1 and m_2 are positive constants. Mesh spacings $h_{x\mu}$, h_x, $h_{y\mu}$ and h_y are defined by

$$h_{x\mu} = \frac{4\varsigma_x}{N_x}, \quad h_x = \frac{2(1 - 2\varsigma_x)}{N_x}, \quad h_{y\mu} = \frac{4\varsigma_y}{N_y}, \quad h_y = \frac{2(1 - 2\varsigma_y)}{N_y}. \tag{7.2}$$

The mesh $\overline{\omega}^{hx}$ is constructed thus: in each of the subintervals $[0, \varsigma_x]$ and $[1 - \varsigma_x, 1]$ the fine mesh spacing is $h_{x\mu}$ while in the interval $[\varsigma_x, 1 - \varsigma_x]$ the coarse mesh spacing is h_x. The mesh $\overline{\omega}^{hy}$ is defined similarly. The difference scheme (3.1) on the piecewise uniform mesh (7.1), (7.2) converges μ-uniformly to the solution of the continuous problem (1.1):

$$\max_{t_k \in \overline{\omega}^\tau} \|U(\cdot, t_k) - u(\cdot, t_k)\|_{\overline{\omega}^h} \leq C\left(N^{-1} \ln N + |\theta - 0.5|\tau + \tau^2\right), \tag{7.3}$$

where $N = \min\{N_x, N_y\}$, $\tau = \max_{k \geq 1} \tau_k$, and constant C is independent of μ, N and τ (see [1] for details).

In the following theorems, we prove uniform convergence of the sequences, generated by the proposed monotone iterative methods, to the solution of problem (1.1).

Theorem 7.1. *Let* (6.3) *hold true. For the sequence* $\{V_1^{(n)}\}$, *generated by* (3.2)–(3.4), (6.1), *the following estimate holds:*

$$\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot, t_k) - u(\cdot, t_k)\|_{\overline{\omega}^h} \leq C\left(\delta + N^{-1}\ln N + |\theta - 0.5|\tau + \tau^2\right),$$

$$N = \min\{N_x, N_y\}, \quad \tau = \max_{k \geq 1} \tau_k,$$

where $u(x, y, t_k)$ *is the solution to* (1.1) *and constant* C *is independent of* μ, N *and* τ.

Proof. The proof follows from Theorem 6.1 and (7.3). ☐

Theorem 7.2. *Let the assumptions* (6.3) *and* (6.6) *be satisfied. Then for the sequence* $\{V_1^{(n)}\}$ *generated by* (5.6), (3.3) *and* (3.4) *and* $n_* =$ *fixed, the following estimate holds:*

$$\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot, t_k) - u(\cdot, t_k)\|_{\overline{\omega}^h}] \leq C(q^{n_*-1} + N^{-1}\ln N + |\theta - 0.5|\tau + \tau^2),$$

$$N = \min\{N_x, N_y\}, \quad \tau = \max_{k \geq 1} \tau_k,$$

where $u(x, y, t_k)$ *is the solution to* (1.1), q *is defined in* (6.9), *constant* C *is independent of* μ, N *and* τ, *and the number of iterative steps on each time level* $n_* \geq 2$.

Proof. The proof follows from Theorem 6.2 and (7.3). ☐

8. Numerical Experiments

In this section, we compare convergence properties of the monotone iterative method (3.2)–(3.4) and the monotone iterative method from [3]. The monotone iterative method from [3] is constructed in the assumption that

$$0 \leq f_u \leq c^*, \quad c^* = \text{const} > 0. \tag{8.1}$$

This method utilizes c^* instead of $c^{(n)}(p, t_k)$ in (3.3).

It is found that in all the numerical experiments the basic feature of monotone convergence of upper and lower sequences is observed. In fact, the monotone property of the sequences holds at every mesh point in the domain. This is, of course, to be expected from our theoretical analysis in Theorem 3.1.

As a test problem for (1.1), we consider the reaction-diffusion problem

$$u_t - \mu^2(u_{xx} + u_{yy}) + \frac{u-4}{5-u} = 0, \quad (x, y, t) \in \omega \times (0, T], \tag{8.2a}$$

$$u(x, y, t) = 1, \quad (x, y, t) \in \partial\omega \times (0, T], \tag{8.2b}$$

$$u(x, y, 0) = 0, \quad (x, y) \in \overline{\omega}, \tag{8.2c}$$

where $\omega = \{0 < x < 1\} \times \{0 < y < 1\}$. The steady state solution to the reduced problem ($\mu = 0$) is $u_r = 4$. For $\mu \ll 1$ the problem is singularly perturbed and characterized by boundary layers of width $\mathcal{O}(\mu|\ln\mu|)$ near $\partial\omega$. The steady state solution increases sharply from $u = 1$ on $\partial\omega$ to $u = 4$ on the interior, and the solution to the parabolic problem approaches this steady state with time (see [1] for details). Since $f_u = 1/(5-u)^2 > 0$, condition (1.2) is satisfied.

For the model problem (8.2), we solve the nonlinear difference scheme (3.1) with the monotone iterative method (3.2)–(3.4), (6.1). The mesh function $V_1^{(0)}(p, t_1)$ defined by

$$V_1^{(0)}(\omega^h, t_1) = 4, \quad V_1^{(0)}(\partial \omega^h, t_1) = 1, \tag{8.3}$$

is an upper solution with respect to the initial condition $g(\omega^h, 0) = 0$, $g(\partial \omega^h, 0) = 1$. We initiate the iterative method with $V_1^{(0)}(p, t_1)$ and thus generate a sequence of upper solutions. At the next time level, t_{k+1} $k \geq 1$, we require an initial iterate that is an upper solution with respect to $V_1(p, t_k)$. Since the boundary condition and function $f(u) = (u - 4)/(5 - u)$ are independent of time, we may choose $V_1^{(0)}(p, t_{k+1}) = V_1(p, t_k)$, $p \in \overline{\omega}^h$. Now, from Theorem 3.1, it follows by induction on k that the mesh function $V_{-1}(p, t_{k+1})$ defined by $V_{-1}(\omega^h, t_{k+1}) = 0$, $V_{-1}(\partial \omega^h, t_{k+1}) = 1$ is a lower solution with respect to $V_1(p, t_k)$ and thus our computed mesh functions satisfy

$$0 \leq V_1^{(n)}(p, t_k) \leq 4, \quad p \in \overline{\omega}^h, \quad 0 \leq n \leq n_*, \quad 0 \leq k \leq N_\tau. \tag{8.4}$$

From here and $f_{uu} = 2/(5 - u)^3$, we conclude (5.4), and, hence, $c^{(n)}(p, t_k)$ is defined by (5.5).

From (8.4), we can also conclude that $f_u = 1/(5 - u)^2$ is bounded below and above by $c_* = 1/25$ and $c^* = 1$, respectively. Thus, in the monotone iterative method from [3], $c^* = 1$ is in use.

We take as our convergence tolerance $\delta = 10^{-5}$ in (6.1). All the discrete linear systems are solved by the ICCG-solver [16]. In our numerical experiments, we use $N_x = N_y = N$, $\tau_k = \tau$, $k \geq 1$.

Table 8.1: Average convergence iteration counts for problem (8.2). The results, corresponding to the monotone iterative method (3.2)–(3.4), (6.1) and monotone iterative method from [3], are given above and below the line, respectively.

N	16	32	64	128	256
μ			$\tau = 0.5$		
1	$\frac{4.0}{7.2}$	$\frac{4.0}{7.2}$	$\frac{4.0}{7.2}$	$\frac{4.0}{7.2}$	$\frac{4.0}{7.2}$
10^{-1}	$\frac{4.1}{8.0}$	$\frac{4.1}{8.0}$	$\frac{4.1}{8.0}$	$\frac{4.1}{8.0}$	$\frac{4.1}{8.0}$
$\leq 10^{-2}$	$\frac{4.1}{7.7}$	$\frac{4.1}{7.7}$	$\frac{4.1}{7.9}$	$\frac{4.1}{8}$	$\frac{4.1}{8}$
μ			$\tau = 0.1$		
1	$\frac{3.4}{5.6}$	$\frac{3.6}{5.6}$	$\frac{3.6}{5.6}$	$\frac{4.2}{5.6}$	$\frac{4.2}{5.6}$
10^{-1}	$\frac{3.2}{6.1}$	$\frac{3.2}{6.1}$	$\frac{4.0}{6.1}$	$\frac{4.0}{6.1}$	$\frac{4.0}{6.1}$
$\leq 10^{-2}$	$\frac{3.4}{6.1}$	$\frac{3.7}{6.1}$	$\frac{4.0}{6.1}$	$\frac{4.0}{6.1}$	$\frac{4.0}{6.1}$

In Table 8.1, for $\tau = 0.5, 0.1$ and for various values of μ and N, we give the average (over ten time levels) convergence iteration counts in the case of the weighting parameter $\theta = 1/2$ (the Crank-Nicolson scheme [14]). The results, corresponding to the monotone iterative method (3.2)–(3.4), (6.1) and monotone iterative method from [3] with $\theta = 1/2$ are given above and below the line, respectively. From the numerical data, it follows that for all values of τ, N and μ the monotone iterative method (3.2)–(3.4), (6.1) faster than the corresponding monotone iterative method from [3]. For τ and N fixed and $\mu \leq 10^{-2}$, the average convergence iteration counts are uniform with respect to μ.

In Table 8.2, for $\tau = 0.1$ and for various values of μ and N, we give the ratio $q_{n+1}(t_k)/[q_n(t_k)]^2$, $k \geq 3$, in the case of the weighting parameter $\theta = 1/2$, where $q_n(t_k)$ is defined in (5.11). From

Table 8.2:   The ratio $q_{n+1}(t_k)/[q_n(t_k)]^2$, $k \geq 3$.

N	32	64	128	256
n	$\mu = 10^{-1}$			
1	2.14×10^{-2}	2.21×10^{-2}	2.34×10^{-2}	2.47×10^{-2}
2	1.97×10^{-2}	2.03×10^{-2}	2.15×10^{-2}	2.28×10^{-2}
3	na	2.01×10^{-2}	2.13×10^{-2}	2.23×10^{-2}
n	$\mu \leq 10^{-2}$			
1	4.09×10^{-2}	4.51×10^{-2}	4.78×10^{-2}	5.23×10^{-2}
2	3.45×10^{-2}	3.60×10^{-2}	4.21×10^{-2}	4.55×10^{-2}
3	3.34×10^{-2}	3.57×10^{-2}	3.80×10^{-2}	4.41×10^{-2}

the numerical data, it follows that for all values of N and μ the monotone iterative method (3.2)–(3.4), (6.1) converges quadratically and for N fixed, $\mu \leq 10^{-2}$ converges uniformly in the perturbation parameter μ. These numerical experiments confirm the theoretical results from Theorems 5.1 and 5.2.

We draw the following conclusions from the numerical experiments:

- The monotone iterative method (3.2)–(3.4), (6.1) converges faster than the corresponding monotone iterative method from [3].

- For τ and the diffusion coefficient μ fixed, the average convergence iteration counts increase slightly with increasing N or independent of N.

- For $\mu \leq 10^{-2}$, the average convergence iteration counts are uniform with respect to μ.

- On time levels, the monotone iterative method (3.2)–(3.4), (6.1) converges quadratically and for $\mu \leq 10^{-2}$ uniformly in μ.

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