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NUMERICAL RECONSTRUCTION OF SMALL PERTURBATIONS IN THE ELECTROMAGNETIC COEFFICIENTS OF A DIELECTRIC MATERIAL*

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Abstract

The aim of this paper is to solve numerically the inverse problem of reconstructing small amplitude perturbations in the magnetic permeability of a dielectric material from partial or total dynamic boundary measurements. Our numerical algorithm is based on the resolution of the time-dependent Maxwell equations, an exact controllability method and Fourier inversion for localizing the perturbations. Two-dimensional numerical experiments illustrate the performance of the reconstruction method for different configurations even in the case of limited-view data.

Mathematics subject classification: (MSC2010) 35Q61, 78M10, 35R30, 65M32, 65M60.
Key words: Maxwell's equations, Inverse problems, Numerical reconstruction, Exact boundary controllability, Fourier inversion.

1. Introduction and Presentation of the Inverse Problem

Inverse problems arise naturally in various areas of science and engineering and have many applications, e. g. in medical imaging, nondestructive testing or underground prospection. The control of the welding integrity of materials for example, is of the utmost importance for aeronautics and nuclear power safety. Several analytical and numerical studies have been devoted to the detection of inhomogeneities in the conductivity, or more generally, in the electromagnetic parameters of a body (see for example [2, 3, 7, 9, 11, 16, 29]). The localization procedure combines an asymptotic formula for the perturbation of the electromagnetic field with an inversion algorithm. The underlying direct problem is in general stationary or harmonic in time. In the present paper we consider such an approach for the resolution of a *time-dependent* electromagnetic inverse problem.

We focus on the numerical reconstruction of small amplitude perturbations in the electromagnetic parameters of a dielectric material from dynamic measurements on a part of the boundary. These problems appear typically in nondestructive testing and quality control of nonmetallic structures, for instance in the construction sector. In this context, reconstruction methods that allow partial boundary data are very interesting because, in most experimental settings, one does not have access to measurements on the whole boundary.

We apply an approach derived in [1]. Partial dynamic boundary measurements of the electric field are the - synthetic or experimental - data of the inversion algorithm. An asymptotic formula

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expresses the effect of the small perturbations on these measurements. This yields a constructive numerical method for the localization of electromagnetic defects in the material. Recently, this has been tested successfully in the context of the wave equation for retrieving small conductivity imperfections [10]. The aim of the present paper is to generalize the method to the second order Maxwell equations. We present the algorithm in the case of a two-dimensional test domain. This is a first prospective study to show the effectiveness of the identification procedure. Since the underlying theoretical results have been obtained in three dimensions of space, it should be possible to realize the inspection of three-dimensional objects in the same way, provided one has a robust 3D Maxwell solver.

Let Ω be a bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma := \partial\Omega$. Let $\boldsymbol{n}$ denote the outward unit normal to Γ. Let Ω' be a smooth subdomain of Ω. We assume that Ω is filled with a material of constant electric permittivity $\varepsilon = 1$ and magnetic permeability

$$\mu_\alpha(\boldsymbol{x}) = 1 + \alpha p(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega. \tag{1.1}$$

The function p quantifies the perturbations of the permeability with respect to the homogeneous background medium ($\varepsilon = 1$, $\mu = 1$) and is supposed to satisfy the following conditions

$$p \in \mathcal{C}^1(\overline{\Omega}),\ \ p \equiv 0 \text{ in } \Omega \setminus \overline{\Omega'}, \text{ and } |p(\boldsymbol{x})| \le M,\ \forall \boldsymbol{x} \in \Omega'. \tag{1.2}$$

The parameter $\alpha > 0$ designates the common order of magnitude of the perturbations which are assumed to be small compared to the background. Consequently, we may assume that there is $\underline{\mu} > 0$ such that $\mu_\alpha(\boldsymbol{x}) \ge \underline{\mu}$ for all $\boldsymbol{x} \in \Omega$. The electric charge and current density in Ω are supposed to be zero, and the problem is driven by an impressed source acting on the boundary (or a part of it) and prescribed initial states at $t = 0$.

Let $T > 0$ be the given final time and let $\boldsymbol{E}(\boldsymbol{x}, t)$ denote the electric field at a point $\boldsymbol{x} \in \Omega$ at time $t \in [0, T]$. In the absence of perturbations, the field $\boldsymbol{E}$ is solution to the following second-order system, derived from Maxwell's equations,

$$\begin{cases} \partial_t^2 \boldsymbol{E} + \mathbf{curl}(\operatorname{curl} \boldsymbol{E}) = \mathbf{0}, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{E} = 0, & \text{in } \Omega \times (0, T), \\ \boldsymbol{E} \times \boldsymbol{n} = F, & \text{on } \Gamma \times (0, T), \\ \boldsymbol{E}(0) = \boldsymbol{E}_0,\ \ \partial_t \boldsymbol{E}(0) = \boldsymbol{E}_1, & \text{in } \Omega. \end{cases} \tag{1.3}$$

Here, $\{\boldsymbol{E}_0, \boldsymbol{E}_1\}$ and F are respectively the boundary and initial data for the electric field. Notice that in two dimensions, the vector **curl** operator is defined for a scalar function φ by $\mathbf{curl}\,\varphi = (\partial_2\varphi, -\partial_1\varphi)^T$, whereas the scalar curl operator acting on a vector field $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ is given by $\operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v} = \partial_1 \boldsymbol{v}_2 - \partial_2 \boldsymbol{v}_1$.

Next, consider the electric field $\boldsymbol{E}_\alpha$ in the presence of the perturbations and subject to the same boundary and initial data. $\boldsymbol{E}_\alpha$ is solution to the perturbed problem

$$\begin{cases} \partial_t^2 \boldsymbol{E}_\alpha + \mathbf{curl}(\mu_\alpha^{-1} \operatorname{curl} \boldsymbol{E}_\alpha) = \mathbf{0}, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{E}_\alpha = 0, & \text{in } \Omega \times (0, T), \\ \boldsymbol{E}_\alpha \times \boldsymbol{n} = F, & \text{on } \Gamma \times (0, T), \\ \boldsymbol{E}_\alpha(0) = \boldsymbol{E}_0,\ \ \partial_t \boldsymbol{E}_\alpha(0) = \boldsymbol{E}_1, & \text{in } \Omega. \end{cases} \tag{1.4}$$

Let us introduce the following functional spaces which are naturally involved in the setting of Maxwell's equations (see, for example, [25]):

$$J = \left\{ \boldsymbol{f} \in \left(L^2(\Omega)\right)^2 \mid \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega \right\},$$