Journal of Computational Mathematics
Vol.32, No.2, 2014, 107–135.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1401-m4357

MIXED INTERIOR PENALTY DISCONTINUOUS GALERKIN METHODS FOR ONE-DIMENSIONAL FULLY NONLINEAR SECOND ORDER ELLIPTIC AND PARABOLIC EQUATIONS*

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Abstract

This paper is concerned with developing accurate and efficient numerical methods for one-dimensional fully nonlinear second order elliptic and parabolic partial differential equations (PDEs). In the paper we present a general framework for constructing high order interior penalty discontinuous Galerkin (IP-DG) methods for approximating viscosity solutions of these fully nonlinear PDEs. In order to capture discontinuities of the second order derivative u_{xx} of the solution u, three independent functions p_1, p_2 and p_3 are introduced to represent numerical derivatives using various one-sided limits. The proposed DG framework, which is based on a nonstandard mixed formulation of the underlying PDE, embeds a nonlinear problem into a mostly linear system of equations where the nonlinearity has been modified to include multiple values of the second order derivative u_{xx}. The proposed framework extends a companion finite difference framework developed by the authors in [9] and allows for the approximation of fully nonlinear PDEs using high order polynomials and non-uniform meshes. In addition to the nonstandard mixed formulation setting, another main idea is to replace the fully nonlinear differential operator by a numerical operator which is consistent with the differential operator and satisfies certain monotonicity (called g-monotonicity) properties. To ensure such a g-monotonicity, the crux of the construction is to introduce the numerical moment, which plays a critical role in the proposed DG framework. The g-monotonicity gives the DG methods the ability to select the mathematically "correct" solution (i.e., the viscosity solution) among all possible solutions. Moreover, the g-monotonicity allows for the possible development of more efficient nonlinear solvers as the special nonlinearity of the algebraic systems can be explored to decouple the equations. This paper also presents and analyzes numerical results for several numerical test problems which are used to guage the accuracy and efficiency of the proposed DG methods.

Mathematics subject classification: 65N30, 65M60, 35J60, 35K55.
Key words: Fully nonlinear PDEs, Viscosity solutions, Discontinuous Galerkin methods.

1. Introduction

Fully nonlinear partial differential equations (PDEs) refer to a class nonlinear PDEs which is nonlinear in the highest order derivatives of the unknown functions in the equations. Due

* Received January 11,2013 / Revised version received December 23, 2013 / Accepted January 15, 2014 / Published online March 31, 2014 /

to their strong nonlinearity, this class of PDEs are most difficult to analyze analytically and to approximate numerically. In the mean time, fully nonlinear PDEs arise in many applications such as antenna design, astrophysics, differential geometry, fluid mechanics, image processing, meteorology, mesh generation, optimal control, optimal mass transport, etc [8], which calls for the development of efficient and reliable numerical methods for solving their underlying fully nonlinear PDE problems.

This is the second paper in a series [9] which is devoted to developing finite difference (FD) and discontinuous Galerkin (DG) methods for approximating *viscosity solutions* of the following general one-dimensional fully nonlinear second order elliptic and parabolic equations:

$$F\left(u_{xx}, u_x, u, x\right) = 0, \qquad x \in \Omega := (a, b), \tag{1.1}$$

and

$$u_t + F\left(u_{xx}, u_x, u, t, x\right) = 0, \qquad (x, t) \in \Omega_T := \Omega \times (0, T], \tag{1.2}$$

which are complemented by appropriate boundary and initial conditions. The *goal* of this paper is to design and implement a class of interior penalty discontinuous Galerkin (IP-DG) methods which is based on a nonstandard mixed formulation; the proposed IP-DG methods are named mIP-DG methods. For the ease of presenting the main ideas and avoiding the technicalities, in this paper we confine our attention to the *one dimensional* fully nonlinear second order PDE problem. The generalization and extension to the high dimensional case of the mIP-DG methods of this paper will be presented in a forthcoming work [11]. In fact, it will be seen later that even in the one dimensional case, the construction and analysis of the proposed mIP-DG methods is already quite complicated.

It is well known [8] that the primary challenges for approximating viscosity solutions of fully nonlinear PDEs are caused by the very notion of viscosity solutions themselves (see section 2 for the definition). Unlike the notion of weak solutions for linear and quasilinear PDEs, the notion of viscosity solutions by design is non-variational, and, in general, viscosity solutions do not satisfy the underlying PDEs in a tangible sense. The non-variational nature of viscosity solutions immediately prevents any attempt to directly and straightforwardly construct Galerkin-type (including DG) methods for approximating fully nonlinear PDEs; in other words, nonlinearity in the highest order derivatives of the unknown function does not allow one to perform integration by parts to transfer one order of derivatives to test functions as often done with linear and quasilinear PDEs. Another big challenge for approximating viscosity solutions of fully nonlinear PDEs is caused by the conditional uniqueness of viscosity solutions; namely, viscosity solutions may only be unique in a restricted function class. Requiring numerical solutions to stay or approximately stay in the same function class often imposes a difficult constraint for designing numerical methods. Finally, we like to mention that as expected, solving the resulting strongly nonlinear (algebraic) systems, regardless which discretization method is used, is another difficult issue encountered with numerical fully nonlinear PDEs.

The mIP-DG methods proposed in this paper aim to approximate viscosity solutions of (1.1) and (1.2) which belong to $H^1(\Omega)$ in the spatial variable. We note that such a viscosity still does not satisfy the underlying PDEs in a tangible sense. We also mention that in order to approximate viscosity solutions that do not have H^1 regularity in the spatial variable, we refer the reader to a companion paper [12] in which we propose another class of more complicated mixed discontinuous Galerkin that incorporates a local discontinuous Galerkin (LDG) approach instead of the IP-DG approach. Such an alternate LDG approach is also more appropriate when a more accurate approximation for u_x is desired.