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CONVERGENCE AND SUPERCONVERGENCE ANALYSIS OF LAGRANGE RECTANGULAR ELEMENTS WITH ANY ORDER ON ARBITRARY RECTANGULAR MESHES*

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Abstract

This paper is to study the convergence and superconvergence of rectangular finite elements under anisotropic meshes. By using of the orthogonal expansion method, an anisotropic Lagrange interpolation is presented. The family of Lagrange rectangular elements with all the possible shape function spaces are considered, which cover the Intermediate families, Tensor-product families and Serendipity families. It is shown that the *anisotropic interpolation error estimates* hold for *any order Sobolev norm*. We extend the convergence and superconvergence result of rectangular finite elements to *arbitrary rectangular meshes* in a unified way.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50.
Key words: Lagrange interpolation, Anisotropic error bounds, Arbitrary rectangular meshes, Orthogonal expansion, Superconvergence.

1. Introduction

The *nondegenerate assumption* or *regular assumption* on the meshes is a basic condition in the classical convergence analysis of the finite element methods, see [9,16]. Consider a bounded convex domain $\Omega \subset R^2$. Let $\mathcal{J}_h$ be a family of meshes of Ω. Denote the diameter of an element K and the diameter of the inscribed circle of K by h_K and ρ_K, respectively, $h = \max_{K\in\mathcal{J}_h} h_K$. It is assumed in the classical finite element theory that

$$\frac{h_K}{\rho_K} \le C, \quad \forall \ K \in \mathcal{J}_h, \tag{1.1}$$

where C is a positive constant independent of K and the function considered.

We will consider the error estimates of finite elements in degenerate meshes. Then the regular assumption is no longer valid in this case. Conversely, *degenerate elements* (or *anisotropic elements*) are characterized by $\frac{h_K}{\rho_K} \to \infty$ as $h \to 0$. Error estimates for degenerate elements can go back to the works by Babuška and Aziz [6] and by Jamet [19]. Especially, anisotropic

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interpolation error estimates of Lagrange elements have been proved by several authors with different methods, interested readers are referred to [4, 5, 12–14, 20, 30, 34, 35, 38] and references therein. Especially, anisotropic triangle (tetrahedra) Lagrange finite elements have been extensively studied in the above mentioned references.

As we know, compared with triangular elements, the choice of shape functions of rectangular elements have much more possibilities than them and thus the constructions of rectangular finite elements are in many forms. Up to now, only a few elements with bi$-k$ tensor product polynomial space are treated by the above mentioned references, see e.g., [1,3–5,14,33,39], there are still some gaps in the anisotropic error estimates of the Lagrange rectangular elements. In fact, there are three popular polynomial bases for rectangular meshes in engineering practice: (1) Tensor-product spaces; (2) Serendipity families; (3) Intermediate elements. Most of these elements are still missing in anisotropic error estimates.

On the other hand, superconvergence in finite element methods has been highlighted for more than thirty years both in the engineering and scientific computing applications. It is a powerful tool in improving the accuracy, and plays an important role in the a posteriori error estimate, mesh refinement and adaptivity, see and references therein. For the literature, interested readers can refer to the recent works [8, 15, 18, 21–23, 28, 36, 41, 42], the books [7, 11, 24, 27, 43, 44] and the references therein. Note that all the superconvergence results obtained in the above references are studied on regular meshes. There are few superconvergence results are obtained on anisotropic meshes, see [26,31,32,39,40] etc. The problems left to us is: *Do the superconvergence results of rectangular finite elements with arbitrary order still hold under anisotropic meshes?*

In this paper, we show that the interpolation operators of Lagrange rectangular elements obtained by orthogonal expansions satisfy the anisotropic properties, and prove their anisotropic error estimates *on arbitrary rectangular meshes.* The analysis covers the Intermediate families, Tensor-product families and Serendipity families in a unified way. Furthermore, the superapproximation results between the interpolations and the finite element solutions are obtained for the Intermediate families, Tensor-product families on *arbitrary rectangular meshes.*

Lastly, we recall some notations and terminology (or refer to [10, 16]). For a bounded Lipschitz domain $\Omega \subset R^2$, let $(\cdot,\cdot)$ denote the usual L^2 -inner product and $\|u\|_{r,p,\Omega}$ (resp. $|u|_{r,p,\Omega}$) be the usual norm (resp. semi-norm) for the Sobolev space $W^{r,p}(\Omega)$, which are defined as

$$\|v\|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha|\le m} |D^\alpha v|^p\right)^{\frac{1}{p}}, \quad |v|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha|=m} |D^\alpha v|^p\right)^{\frac{1}{p}}.$$

When $p=2$, denote $W^{2,r}(\Omega)$ by $H^r(\Omega)$. We shall also denote by $P_l(G)$ the space of polynomials on G of degrees no more than l. Throughout this paper, C is a positive constant independent of the mesh diameter, $\frac{h_K}{\rho_K}$ and the function considered.

2. Anisotropic Interpolations via Orthogonal Expansions

In this paper, we consider the following large family of rectangular elements defined on the reference square $\widehat{K} = \{\widehat{x} = (\xi,\eta)^T \in \mathbb{R}^2 : -1 < \xi, \eta < 1\}$:

$$\mathcal{Q}_m(k) = \text{Span}\{\xi^i \eta^j : (i,j) \in I_{k,m}, 1 \le m \le k\}, \tag{2.1}$$

$I_{k,m}$ is an index set satisfies

$$\{(i,j)|0 \le i,j \le k, i+j \le m+k\} \subset I_{k,m} \subset \{(i,j)|0 \le i,j \le k\}. \tag{2.2}$$