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A TRIANGULAR FINITE VOLUME ELEMENT METHOD FOR A SEMILINEAR ELLIPTIC EQUATION*

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Abstract

In this paper we extend the idea of interpolated coefficients for a semilinear problem to the triangular finite volume element method. We first introduce triangular finite volume element method with interpolated coefficients for a boundary value problem of semilinear elliptic equation. We then derive convergence estimate in H^1-norm, L^2-norm and L^∞-norm, respectively. Finally an example is given to illustrate the effectiveness of the proposed method.

Mathematics subject classification: 49J20, 65N30.
Key words: Semilinear elliptic equation, Triangulation, Finite volume element with interpolated coefficients.

1. Introduction

The finite volume element method is a discretization technique for partial differential equations, especially for those that arise from physical laws including mass, momentum, and energy. The finite volume element method uses a volume integral formulation of the differential equation with a finite partitioning set of volume to discretize the equation, then restricts the admissible functions to a linear finite element space to discretize the solution [2, 5–7, 19, 20, 22, 23, 25, 26, 29, 30, 33–36, 41]. The method has been widely used in computational fluid mechanics as it preserves the mass conservation. As far as the method is concerned, it is identical to the special case of the generalized difference method or GDM proposed by Li-Chen-Wu [29].

Many works have been devoted to the analysis of finite element methods. see, e.g., [11-18]. For semi-linear problems, the finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal [42]. Later Larsson-Thomee-Zhang [27] studied the semidiscrete linear triangular finite element with interpolated coefficients and Chen-Larsson-Zhang [10] derived almost optimal order convergence on piecewise uniform triangular meshes by the superconvergence techniques. Xiong-Chen studied superconvergence of triangular quadratic finite element and superconvergence of rectangular finite element for semilinear elliptic problem, respectively, and illustrated the effectiveness of the proposed method in some examples [37–39]. Recently Xiong-Chen first put the interpolation idea into the finite volume element method and studied the finite volume element with interpolated coefficients of the two-point boundary problem [40].

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Li [28] considered the finite volume element method for a nonlinear elliptic problem and obtained the error estimate in H^1-norm. Chatzipantelidis-Ginting-Lazarov [8] studied the finite volume element method for a nonlinear elliptic problem, and established the error estimates in H^1-norm, L^2-norm and L^∞-norm. Bi [3] obtained the H^1 and $W^{1,\infty}$ superconvergence estimates between the solution of the finite volume element method and that of the finite element method for a nonlinear elliptic problem. In this paper, we shall put the excellent interpolating coefficients idea into the finite volume element method on triangular mesh for a semilinear elliptic equation.

We shall denote Sobolev space and its norm by $W^{m,p}(\Omega)$ and $\|\cdot\|_{m,p}$, respectively [1]. If $p=2$, simply use $H^m(\cdot)$ and $\|\cdot\|_m$ and $\|\cdot\|=\|\cdot\|_0$ is L^2-norm. Further we shall denote by p' the adjoint of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1,\ p\geq 1$. We shall assume that the exact solution u is sufficiently smooth for our purpose. The constants C, C_1, C_2, etc. are generic in the paper.

The rest of the paper is organized as follow. First we will introduce the triangular finite volume element method with interpolated coefficients in Section 2 and give preliminaries and some lemmas in Section 3. Next we derive optimal order H^1-norm, L^2-norm and L^∞-norm estimates, respectively, in Section 4. Finally the theoretical results are tested by a numerical example in Section 5.

2. Finite Volume Element Method with Interpolated Coefficients

Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal domain. Consider the second-order semilinear elliptic boundary value problem:

$$\begin{cases} -\frac{\partial}{\partial x}\Big(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\Big)-\frac{\partial}{\partial y}\Big(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\Big)+f(u)=g, \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the coefficients $a_{ij}(x,y)(i,j=1,2)$ are sufficiently smooth functions satisfying the elliptic condition, i.e., there exists a constant $C>0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j \geq C(\xi_1^2+\xi_2^2),$$

holds for any real vector $(\xi_1,\xi_2)\in\mathbb{R}^2$ and $(x,y)\in\overline{\Omega}$. It is also assumed that $f'(s)>0$ for $s\in(-\infty,+\infty)$ and $f''(s)$ is continuous with respect to s.

Let $V\subset\Omega$ be any control volume with piecewise smooth boundary ∂V. Integrate (2.1) over control volume V, then by the Green's formula, the conservative integral of (2.1) reads, finding u, such that

$$-\int_{\partial V} W^{(1)}\mathrm{d}y+\int_{\partial V} W^{(2)}\mathrm{d}x+\int_V f(u)\mathrm{d}x\mathrm{d}y=\int_V g\mathrm{d}x\mathrm{d}y, \quad V\subset\Omega, \tag{2.2}$$

where

$$W^{(i)}=a_{i1}\frac{\partial u}{\partial x}+a_{i2}\frac{\partial u}{\partial y}, \quad i=1,2.$$

In this paper, we shall consider triangular partition of Ω and piecewise triangle linear interpolation with interpolated coefficients, for u.

Give a quasi-uniform triangulation $\mathcal{J}_h$ for Ω with $h=\max h_K$, where h_K is the diameter of the triangle $K\in\mathcal{J}_h$. All control volumes are constructed in the following way. Let Q_K be the barycentre of $K \in \mathcal{J}_h$. Connect Q_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals $K_P, P \in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $P \in Z_h = \cup_{K\in\mathcal{J}_h} Z_h(K)$ we associate a control volume V_P, which consists of the subregions K_P, sharing the vertex P_0 (see Fig. 2.1). Denote the set of interior vertices of Z_h by Z_h^0. For boundary nodes, their control volumes should be modified correspondingly. All the control volumes constitute the dual partition $\mathcal{J}_h^*$.

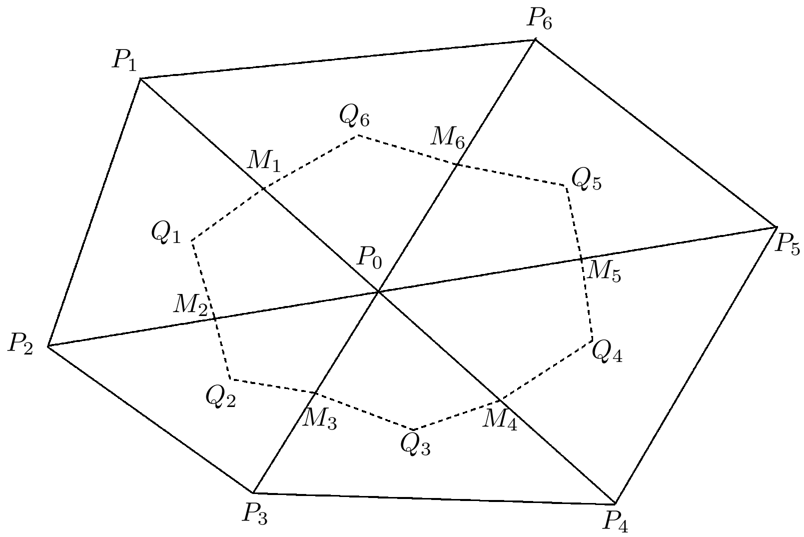

Fig. 2.1. Illustration for a dual element V_{P_0} and its modes.

Let $S_h \subset H^1(\Omega)$ and $S_{0h} \subset H_0^1(\Omega)$ be both the piecewise triangular linear finite element subspace over the partition $\mathcal{J}_h$, and S_h^* be the piecewise constant space over the dual partition $\mathcal{J}_h^*$. Define interpolation operator $\mathrm{I}_h : C(\Omega) \to S_h$ and interpolation operator $\mathrm{I}_h^* : C(\Omega) \to S_h^*$. For an arbitrary node $P \in Z_h^0$, denote φ_P by nodal basic function of P and χ_P by characteristic function over V_P, then we have

$$\mathrm{I}_h v = \sum_{P\in Z_h^0} v(P)\varphi_P, \quad \forall v \in C(\Omega), \tag{2.3}$$

$$\mathrm{I}_h^* v = \sum_{P\in Z_h^0} v(P)\chi_P, \quad \forall v \in C(\Omega). \tag{2.4}$$

The standard finite volume element scheme of (2.2) can read, finding $\bar{u}_h \in S_{0h}$, such that

$$-\int_{\partial V_{P_0}} \overline{W}_h^{(1)} \mathrm{d}y + \int_{\partial V_{P_0}} \overline{W}_h^{(2)} \mathrm{d}x + \int_{V_{P_0}} f(\bar{u}_h) \mathrm{d}x\mathrm{d}y = \int_{V_{P_0}} g \mathrm{d}x\mathrm{d}y, \quad \forall P_0 \in Z_h^0,$$

where

$$\overline{W}_h^{(i)} = a_{i1}\frac{\partial \bar{u}_h}{\partial x} + a_{i2}\frac{\partial \bar{u}_h}{\partial y}, \quad i = 1, 2.$$

For the sake of simplicity, we now define triangular linear finite volume element scheme with interpolated coefficients, finding $u_h \in S_{0h}$, such that

$$-\int_{\partial V_{P_0}} W_h^{(1)} \mathrm{d}y + \int_{\partial V_{P_0}} W_h^{(2)} \mathrm{d}x + \int_{V_{P_0}} \mathrm{I}_h f(u_h) \mathrm{d}x\mathrm{d}y = \int_{V_{P_0}} g \mathrm{d}x\mathrm{d}y, \quad \forall P_0 \in Z_h^0, \tag{2.5}$$

where

$$W_h^{(i)} = a_{i1}\frac{\partial u_h}{\partial x} + a_{i2}\frac{\partial u_h}{\partial y}, \quad i = 1, 2.$$

Eq. (2.5) can be further written as difference equations which is simpler than that of the standard finite volume element method. It can be solved by the Newton iteration method in which its tangent matrix can be calculated in a simple way.

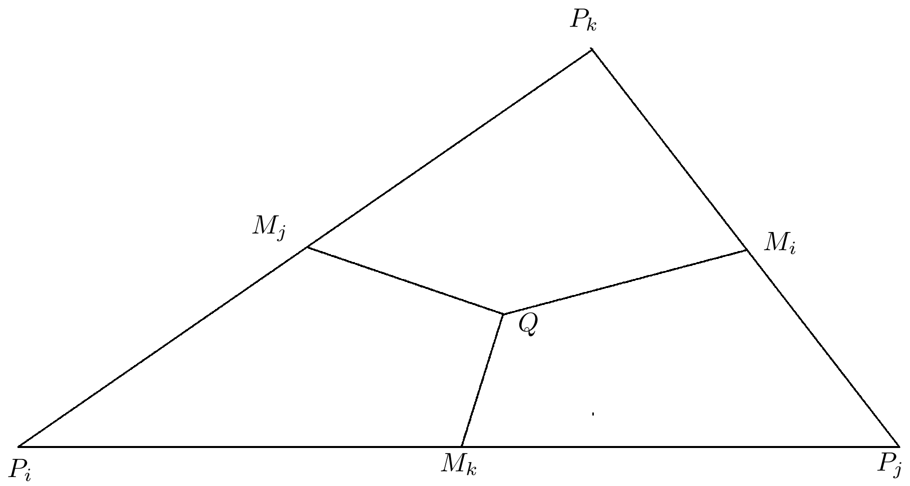

Fig. 2.2. A triangle K partitioned into the three subregions.

Let $K = \triangle P_iP_jP_k$ be any triangle and $P(x, y)$ a point in the triangle. As an example we take $A = -\Delta$, then (2.5) becomes

$$-\int_{\partial V_{P_0}} \frac{\partial u_h}{\partial n}\mathrm{d}s + \int_{V_{P_0}} \mathrm{I}_h f(u_h)\mathrm{d}x\mathrm{d}y = \int_{V_{P_0}} g\mathrm{d}x\mathrm{d}y, \quad \forall P_0 \in Z_h^0.$$

For a triangle $K_{Q_i} = \triangle P_0P_iP_{i+1}$, denote $a_i = \overline{P_{i+1}P_0}, b_i = \overline{P_iP_0}$ and $c_i = \overline{P_iP_{i+1}}$ (see [29]), where $P_7 = P_1$. Then we can get

$$\sum_{i=1}^{6} \frac{1}{4S_{Q_i}}\Big[(u_{P_i} - u_{P_0})\frac{b_i^2 - c_i^2 - a_i^2}{2} + (u_{P_{i+1}} - u_{P_0})\frac{a_i^2 - b_i^2 - c_i^2}{2}\Big]$$
$$+\sum_{i=1}^{6} \frac{S_{Q_i}}{108}(22f_{P_0} + 7f_{P_i} + 7f_{P_{i+1}}) = \int_{V_{P_0}} g\mathrm{d}x\mathrm{d}y, \quad \forall P_0 \in Z_h^0, \tag{2.6}$$

where S_{Q_i} is the area of the triangle $K_{Q_i} = \triangle P_0P_iP_{i+1}$ and $u_{P_i} = u_h(P_i), f_{P_i} = f(u_h(P_i))$. Obviously (2.6) is a nonlinear system with respect to u_{P_i}. For nonregular inner nodes (x_{P_i}, y_{P_i}), by boundary condition the above equation should be modified correspondingly.

3. Preliminaries and Lemmas

In the preceding section, we give the finite volume element scheme with interpolated coefficients. We will give preliminary work and some lemmas in this section. Let

$$a(u,\mathrm{I}_h^*\varphi_h)=\sum_{P\in Z_h^0}\varphi_h(P)\Big(-\int_{\partial V_P}W^{(1)}\mathrm{d}y+\int_{\partial V_P}W^{(2)}\mathrm{d}x\Big),\quad \forall\varphi_h\in S_{0h},$$

$$(u,\mathrm{I}_h^*\varphi_h)=\sum_{P\in Z_h^0}\varphi_h(P)\int_{V_P}u\mathrm{d}x\mathrm{d}y,\quad \forall\varphi_h\in S_{0h},$$

and take $V=V_P$. Then (2.2) can be written as, finding $u\in H_0^1(\Omega)$, such that

$$a(u,\mathrm{I}_h^*\varphi_h)+(f(u),\mathrm{I}_h^*\varphi_h)=(g,\mathrm{I}_h^*\varphi_h),\qquad \forall\varphi_h\in S_{0h}. \tag{3.1}$$

Analogously, (2.5) is equivalent to finding $u_h\in S_{0h}$, such that

$$a(u_h,\mathrm{I}_h^*\varphi_h)+(\mathrm{I}_hf(u_h),\mathrm{I}_h^*\varphi_h)=(g,\mathrm{I}_h^*\varphi_h),\qquad \forall\varphi_h\in S_{0h}. \tag{3.2}$$

For the sake of simplicity in our analysis, we still denote the bilinear form by

$$a(u,v)=\int_\Omega\left(W^{(1)}\frac{\partial v}{\partial x}+W^{(2)}\frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y,\quad \forall u,v\in H_0^1(\Omega).$$

Depicted as in Fig. 2.2, we convert the integral on the edge of dual partition to the related element $K=\triangle P_iP_jP_k\in\mathcal{J}_h$. Then

$$\begin{aligned}a(u,\mathrm{I}_h^*\varphi_h)&=\sum_{K\in\mathcal{J}_h}\sum_{l=i,j,k}\varphi_h(P_l)\Big(-\int_{\partial V_{P_l}\cap K}W^{(1)}\mathrm{d}y+\int_{\partial V_{P_l}\cap K}W^{(2)}\mathrm{d}x\Big)\\&=-\sum_{K\in\mathcal{J}_h}\sum_{l=i,j,k}\int_{\partial V_{P_l}\cap K}(W^{(1)},W^{(2)})\cdot n\mathrm{I}_h^*\varphi_h\mathrm{d}s,\quad \forall\varphi_h\in S_{0h}.\end{aligned} \tag{3.3}$$

Similarly we can obtain

$$(u,\mathrm{I}_h^*\varphi_h)=\sum_{K\in\mathcal{J}_h}\int_K u\mathrm{I}_h^*\varphi_h\mathrm{d}x\mathrm{d}y=\sum_{K\in\mathcal{J}_h}\sum_{l=i,j,k}\varphi_h(P_l)\int_{V_{P_l}\cap K}u\mathrm{d}x\mathrm{d}y,\quad \forall\varphi_h\in S_{0h}. \tag{3.4}$$

Denote $\|\cdot\|_s$ and $|\cdot|_s$ be continuous norm and continuous semi-norm of order s in Sobolev space $H^s(\Omega)$, respectively. Define discrete zero norm, semi-norm and full-norm, respectively, by

$$\|\varphi_h\|_{0,h}=\left\{\sum_{K\in\mathcal{J}_h}\|\varphi_h\|_{0,h,K}^2\right\}^{1/2}, \tag{3.5}$$

$$|\varphi_h|_{1,h}=\left\{\sum_{K\in\mathcal{J}_h}|\varphi_h|_{1,h,K}^2\right\}^{1/2}, \tag{3.6}$$

$$\|\varphi_h\|_{1,h}=\left(\|\varphi_h\|_{0,h}^2+|\varphi_h|_{1,h}^2\right)^{1/2}, \tag{3.7}$$

for $\varphi_h\in S_{0h}$, where $K=\triangle P_iP_jP_k$, shown as in Fig 2.2, and

$$|\varphi_h|_{0,h,K}=\left[\frac{1}{3}(\varphi_i^2+\varphi_j^2+\varphi_k^2)S_K\right]^{1/2},$$

$$|\varphi_h|_{1,h,K}=\left\{\left[\left(\frac{\partial\varphi_h(Q)}{\partial x}\right)^2+\left(\frac{\partial\varphi_h(Q)}{\partial y}\right)^2\right]S_K\right\}^{1/2}.$$

From [29], we have the following lemmas.

Lemma 3.1. *For* $\forall\varphi_h \in S_{0h}$, $|\varphi_h|_{1,h}$ *and* $|\varphi_h|_1$ *are identical and* $\|\varphi_h\|_{0,h}$ *and* $\|\varphi_h\|_{1,h}$ *are equivalent with* $\|\varphi_h\|_0$ *and* $\|\varphi_h\|_1$ *respectively, i.e., there exist positive constants* C_1, C_2, C_3, C_4 *independent of* S_{0h} *such that*

$$C_1|\varphi_h|_{0,h} \le |\varphi_h|_0 \le C_2|\varphi_h|_{0,h}, \qquad \forall\varphi_h \in S_h, \tag{3.8}$$

$$C_3\|\varphi_h\|_{1,h} \le \|\varphi_h\|_1 \le C_4\|\varphi_h\|_{1,h} \quad \forall\varphi_h \in S_h. \tag{3.9}$$

From [7, 9, 29], we have the following three lemmas.

Lemma 3.2. ([29]) *There exist positive constants* C_1, C_2 *such that*

$$a(\varphi_h, \mathrm{I}_h^*\varphi_h) \ge C_1|\varphi_h|_1^2, \qquad \forall\varphi_h \in S_{0h}, \tag{3.10}$$

$$|a(u - \mathrm{I}_h u, \mathrm{I}_h^*\varphi_h)| \le C_2 h\|u\|_2|\varphi_h|_1, \quad \forall u \in H_0^1(\Omega),\ \varphi_h \in S_{0h}. \tag{3.11}$$

Lemma 3.3. ([29]) *The semi-norm* $|\cdot|_1$ *and the norm* $\|\cdot\|_1$ *are equivalent in the space* $H_0^1(\Omega)$, *that is, there exists positive constants* C *such that*

$$|\varphi_h|_1 \le \|\varphi_h\|_1 \le C|\varphi_h|_1, \quad \forall\varphi_h \in S_{0h}. \tag{3.12}$$

Lemma 3.4. *The interpolation operator* I_h^* *has the following properties*

$$\int_K \mathrm{I}_h^* v_h \mathrm{d}x\mathrm{d}y = \int_K v_h \mathrm{d}x\mathrm{d}y, \quad \forall v_h \in S_{0h}, \text{for any } K \in \mathcal{J}_h, \tag{3.13}$$

$$\int_e \mathrm{I}_h^* v_h \mathrm{d}s = \int_e v_h \mathrm{d}s, \qquad \forall v_h \in S_{0h},\ \text{for any side of } K \in \mathcal{J}_h, \tag{3.14}$$

$$\|\mathrm{I}_h^* v_h\|_{e,\infty} \le \|v_h\|_{e,\infty}, \quad \forall v_h \in S_{0h},\ \text{for any side of } K \in \mathcal{J}_h, \tag{3.15}$$

$$\|\varphi_h - \mathrm{I}_h^*\varphi_h\|_{0,p,K} \le Ch|\varphi_h|_{1,p,K}, \quad \forall\varphi_h \in S_{0h}, \quad 1 \le p \le \infty. \tag{3.16}$$

Proof. For $v_h \in S_h$ in $K \in \mathcal{J}_h$, write v_h as

$$v_h = v_h(P_i)\lambda_i + v_h(P_j)\lambda_j + v_h(P_k)\lambda_k.$$

Then we have

$$\int_K \mathrm{I}_h^* v_h \mathrm{d}x\mathrm{d}y = \sum_{l=i,j,k} v_h(P_l) \int_{K\cap V_{P_l}} \mathrm{d}x\mathrm{d}y = \frac{1}{3}[v_h(P_i) + v_h(P_j) + v_h(P_k)]S_K,$$

$$\int_K v_h \mathrm{d}x\mathrm{d}y = \sum_{l=i,j,k} \int_K v_h(P_l)\lambda_l \mathrm{d}x\mathrm{d}y = \frac{1}{3}[v_h(P_i) + v_h(P_j) + v_h(P_k)]S_K.$$

The desired result (3.13) is derived from the above two formulations. From [9] we also obtain (3.14)–(3.16). □

For the interpolation operator I_h, we need the following lemma.

Lemma 3.5. ([9]) *Assume* w, φ *are sufficiently smooth functions. Let* $\mathrm{I}_h\varphi \in S_{0h}$ *be the Lagrangian interpolation of* φ, *then*

$$|(w\,(\varphi - \mathrm{I}_h\varphi), \psi_h)| \le Ch^2\|\varphi\|_{2,p}\|\psi_h\|_{1,p'}, \quad \forall\psi_h \in S_{0h}, \tag{3.17}$$

for $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p \le \infty$.

In view of the Schwartz inequality, we can obtain the following result.

Lemma 3.6. *Assume $w \in H_0^1(\Omega)$, then there exists a positive constant C, independent of the mesh size h, such that*

$$|(w - \mathrm{I}_h w, \mathrm{I}_h^* \varphi_h)| \le Ch^2 \|w\|_2 \|\varphi_h\|_0, \quad \forall \varphi_h \in S_{0h}. \tag{3.18}$$

Lemma 3.7. ([8]) *Let e be a side of a triangle $K \in \mathcal{J}_h$. Then for $w \in H^1(K)$ there exists a constant $C > 0$ independent of h such that*

$$\left| \int_e w(v_h - \mathrm{I}_h^* v_h) \mathrm{d}s \right| \le Ch^2 \|u\|_{1,K} \|v_h\|_{1,K}, \quad \forall v_h \in S_h. \tag{3.19}$$

Moreover, for $g \in H^1$ and $v_h \in S_{0h}$,

$$(g, v_h - \mathrm{I}_h^* v_h) \le Ch^2 \|g\|_1 \|v_h\|_1. \tag{3.20}$$

For our theoretical analysis, we also need the following two lemmas.

Lemma 3.8. *Let $u \in H^2$. The following identities hold*

$$\sum_{K \in \mathcal{J}_h} \int_{\partial K} (W^{(1)}, W^{(2)}) \cdot n v_h \mathrm{d}s = 0, \quad \sum_{K \in \mathcal{J}_h} \int_{\partial K} (W^{(1)}, W^{(2)}) \cdot n \mathrm{I}_h^* v_h \mathrm{d}s = 0, \tag{3.21}$$

$$\sum_{K \in \mathcal{J}_h} \int_{\partial K} (W_e^{(1)}, W_e^{(2)}) \cdot n v_h \mathrm{d}s = 0, \quad \sum_{K \in \mathcal{J}_h} \int_{\partial K} (W_e^{(1)}, W_e^{(2)}) \cdot n \mathrm{I}_h^* v_h \mathrm{d}s = 0, \tag{3.22}$$

where $W_e^{(i)} = a_{i1}(e)\frac{\partial u}{\partial x} + a_{i2}(e)\frac{\partial u}{\partial y}, i = 1, 2$ and $a_{ij}(e)$ are the value of a_{ij} at the midpoint of the edge e of triangle $K \in \mathcal{J}_h$.

Proof. The first identity of (3.21) is obvious by rewriting the sum as integrals of jump terms over the interior edges of $\mathcal{J}_h$. These jumps obviously vanish because of the continuity of $(W^{(1)}, W^{(2)}) \cdot n$. A similar argument gives the second identity of (3.21) and two identities of (3.22). □

Lemma 3.9. *Let u_n be defined by (3.2). For any $v_n \in S_h$,*

$$|a(u_h, v_h) - a(u_h, \mathrm{I}_h^* v_h)| \le C\Big(h^2 \|u\|_2 + h\|u - u_h\|_1\Big) \|v_h\|_1. \tag{3.23}$$

Proof. Using the Green's formula, the identity

$$\begin{aligned} &\int_{V_P \cap K} \left(\frac{\partial}{\partial x} W_h^{(1)} + \frac{\partial}{\partial y} W_h^{(2)} \right) \mathrm{d}x\mathrm{d}y \\ &= \int_{V_P \cap \partial K} (W_h^{(1)}, W_h^{(2)}) \cdot n \mathrm{d}s + \int_{\partial V_P \cap K} (W_h^{(1)}, W_h^{(2)}) \cdot n \mathrm{d}s, \end{aligned} \tag{3.24}$$

holds for $P \in Z_h^0$ and $K \in \mathcal{J}_h$. Hence we have

$$\begin{aligned} a(u_h, \mathrm{I}_h^* v_h) = &- \sum_{K \in \mathcal{J}_h} \int_K \left(\frac{\partial}{\partial x} W_h^{(1)} + \frac{\partial}{\partial y} W_h^{(2)} \right) \mathrm{I}_h^* v_h \mathrm{d}x\mathrm{d}y \\ &+ \sum_{K \in \mathcal{J}_h} \int_{\partial K} (W_h^{(1)}, W_h^{(2)}) \cdot n \mathrm{I}_h^* v_h \mathrm{d}s. \end{aligned} \tag{3.25}$$

By the Green's formula, we also obtain

$$\begin{aligned} a(u_h, v_h) &= \sum_{K\in\mathcal{J}_h} \int_K \left(W_h^{(1)} \frac{\partial v_h}{\partial x} + W_h^{(2)} \frac{\partial v_h}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\ &= - \sum_{K\in\mathcal{J}_h} \int_K \left(\frac{\partial}{\partial x} W_h^{(1)} + \frac{\partial}{\partial y} W_h^{(2)} \right) v_h \mathrm{d}x\mathrm{d}y + \sum_{K\in\mathcal{J}_h} \int_{\partial K} (W_h^{(1)}, W_h^{(2)}) \cdot n v_h \mathrm{d}s. \end{aligned} \tag{3.26}$$

Subtracting (3.25) from (3.26) gives

$$\begin{aligned} a(u_h, v_h) - a(u_h, \mathrm{I}_h^* v_h) = &- \sum_{K\in\mathcal{J}_h} \int_K \left(\frac{\partial}{\partial x} W_h^{(1)} + \frac{\partial}{\partial y} W_h^{(2)} \right) (v_h - \mathrm{I}_h^* v_h) \mathrm{d}x\mathrm{d}y \\ &+ \sum_{K\in\mathcal{J}_h} \int_{\partial K} (W_h^{(1)}, W_h^{(2)}) \cdot n (v_h - \mathrm{I}_h^* v) \mathrm{d}s. \end{aligned} \tag{3.27}$$

Lemma 3.8 gives the identity

$$\sum_{K\in\mathcal{J}_h} \int_{\partial K} \Big(- W^{(1)} - (W_h^{(1)} - W^{(1)})_e, -W_h^{(2)} - (W_h^{(2)} - W_h^{(2)})_e \Big) \cdot n (v_h - \mathrm{I}_h^* v) \mathrm{d}s = 0,$$

where

$$(W_h^{(i)} - W^{(i)})_e = a_{i1}(e) \frac{\partial u_h - u}{\partial x} + a_{i2}(e) \frac{\partial u_h - u}{\partial y}, \quad i = 1, 2.$$

Employing this identity, (3.13) in Lemma 3.4, we get

$$\begin{aligned} &a(u_h, v_h) - a(u_h, \mathrm{I}_h^* v_h) \\ = &- \sum_{K\in\mathcal{J}_h} \int_K \left(\frac{\partial}{\partial x} W_h^{(1)} - \xi_1 + \frac{\partial}{\partial y} W_h^{(2)} - \xi_2 \right) (v_h - \mathrm{I}_h^* v_h) \mathrm{d}x\mathrm{d}y \\ &+ \sum_{K\in\mathcal{J}_h} \int_{\partial K} \Big((W_h^{(1)} - W^{(1)}) - (W_h^{(1)} - W^{(1)})_e, (W_h^{(2)} - W_h^{(2)}) - (W_h^{(2)} - W_h^{(2)})_e \Big) \\ &\cdot n (v_h - \mathrm{I}_h^* v) \mathrm{d}s \equiv \sum_{K\in\mathcal{J}_h} (\mathrm{I}_K + \mathrm{II}_K), \end{aligned} \tag{3.28}$$

where ξ_1 and ξ_2 are the mean values of $\frac{\partial}{\partial x} W_h^{(1)}$ and $\frac{\partial}{\partial y} W_h^{(2)}$ over triangle K, respectively. By using the Holder's inequality, we can get

$$\begin{aligned} |\mathrm{I}_K| &\le Ch(|W_h^{(1)}|_{1,K} + |W_h^{(2)}|_{1,K}) h \|v_h\|_{1,K} \le Ch^2 \|u_h\|_{1,K} \|v_h\|_{1,K} \\ &\le Ch^2 \Big(\|u - u_h\|_{1,K} + \|u\|_{1,K} \Big) \|v_h\|_{1,K}. \end{aligned} \tag{3.29}$$

To bound II_K, we have

$$\begin{aligned} |\mathrm{II}_K| &\le Ch \left(\sum_{i=1}^{2} \left| (a_{i1} - a_{i1}(e)) \frac{\partial (u_h - u)}{\partial x} + (a_{i2} - a_{i2}(e)) \frac{\partial (u_h - u)}{\partial y} \right|_{1,K} \right) \|v_h\|_{1,K} \\ &\le Ch \max |a'_{ij}| (\|u - u_h\|_{1,K} + h\|u\|_{2,K}) \|v_h\|_{1,K}. \end{aligned} \tag{3.30}$$

Summing up (3.29) and (3.30) over all triangles, we obtain the desired (3.23). $\square$

4. Error Estimate of the Finite Volume Element

We have given the definition of the finite volume element scheme with interpolated coefficients. Now we analyze the error of the scheme. To start our analysis, we introduce an auxiliary bilinear form

$$A(u;w,\mathrm{I}_h^*\varphi_h)=a(w,\mathrm{I}_h^*\varphi_h)+(f'(u)w,\mathrm{I}_h^*\varphi_h),$$

where u is the exact solution in (2.1). For the auxiliary bilinear form $A(u;\cdot,\cdot)$, we have following positive definite properties.

Lemma 4.1. *For fixed $u\in H_0^1(\Omega)$, $A(u;w_h,\mathrm{I}_h^*w_h)$ is positive definite for sufficiently small h, i.e., there exists a positive constant α, such that*

$$A(u;w_h,\mathrm{I}_h^*w_h)\geq\alpha(u,f)\|w_h\|_1^2,\quad\forall w_h\in S_{0h}. \tag{4.1}$$

Proof. Rewrite $A(u;w_h,\mathrm{I}_h^*w_h)$ as

$$A(u;w_h,\mathrm{I}_h^*w_h)=a(w_h,\mathrm{I}_h^*w_h)+(f'(u)w_h,w_h)-\Big((f'(u)w_h,w_h)-(f'(u)w_h,\mathrm{I}_h^*w_h)\Big). \tag{4.2}$$

Application of Lemma 3.2 and Lemma 3.3 yields

$$a(w_h,\mathrm{I}_h^*w_h)\geq C_1\|w_h\|_1^2. \tag{4.3}$$

Note that $f'(s)>0$ and let $C_2=\inf\limits_{P\in\Omega}f'(u(P))$ for the fixed u. Then we have

$$(f'(u)w_h,w_h)\geq C_2\|w_h\|_0^2\geq 0. \tag{4.4}$$

It follows from (3.13) in Lemma 3.7 that

$$\begin{aligned}&|(f'(u)w_h,w_h)-(f'(u)w_h,\mathrm{I}_h^*w_h)|\\&=\left|\sum_{K\in\mathcal{J}_h}\int_K f'(u)w_h(w_h-\mathrm{I}_h^*w_h)\mathrm{d}x\mathrm{d}y\right|\leq\sum_{K\in\mathcal{J}_h}Ch|f'(u)w_h|_{1,K}h|w_h|_{1,K}\\&\leq\max_\Omega(|f''(u)\nabla u|,|f'(u)|)\sum_{K\in\mathcal{J}_h}Ch^2\|w_h\|_{1,K}^2\leq C_3h^2\|w_h\|_1^2,\end{aligned} \tag{4.5}$$

This, together with (4.3)–(4.5), gives

$$A(u;w_h,\mathrm{I}_h^*w_h)\geq C_1\|w_h\|_1^2-C_3h^2\|w_h\|_1^2=(C_1-C_3h^2)\|w_h\|_1^2,$$

which implies the desired result (4.1) for sufficiently small h. □

Now we state the main result of this section.

Theorem 4.1. *Assume $f'(s)>0, f\in C^2(R), g\in L^2(\Omega)$. Let $u\in H_0^1(\Omega)\cap H^2(\Omega)$ is the solution of* (2.1) *and $\mathcal{J}_h$ is quasi-uniformly triangular partition of domain Ω, then the approximate solution $u_h\in S_{0h}$ of finite volume element method* (2.5) *with interpolated coefficients converges to the exact solution u with the following estimate*

$$\|u-u_h\|_1\leq C(u,f,g)h, \tag{4.6}$$

for sufficiently small h.

Proof. Subtracting (3.2) from (3.1), we obtain the following error equation

$$a(u-u_h, \mathrm{I}_h^*\varphi_h) + (f(u) - \mathrm{I}_h f(u_h), \mathrm{I}_h^*\varphi_h) = 0. \tag{4.7}$$

By expansion in an element $\tau \in \mathcal{J}_h$, we have

$$\begin{aligned} \mathrm{I}_h(f(u)-f(u_h)) &= \sum_j (f(u(P_j)) - f(u_h(P_j))) \\ &= f'(u)(\mathrm{I}_h u - u_h) + \delta_1 \max_\tau |\mathrm{I}_h u - u_h| + \delta_2 \max_\tau |\mathrm{I}_h u - u_h|^2, \end{aligned} \tag{4.8}$$

where

$$\delta_1 = C \max_{P',P''\in\tau} |f'(u(P')) - f'(u(P''))| = \mathcal{O}(h),$$
$$\delta_2 = \frac{1}{2} f''(\xi) = \mathcal{O}(1), \quad |\xi| \le \max_{P\in\Omega} |u(P)|.$$

Substituting (4.8) into (4.7), we find

$$\begin{aligned} &A(u; u_h - \mathrm{I}_h u_h, \mathrm{I}_h^*\varphi_h) \\ &= a(u; u - \mathrm{I}_h u, \mathrm{I}_h^*\varphi_h) + (f(u) - \mathrm{I}_h(u), \mathrm{I}_h^*\varphi_h) + \sum_{\tau\in\mathcal{J}_h} (r, \mathrm{I}_h^*\varphi_h), \end{aligned}$$

where $r = \delta_1 \max_\tau |\mathrm{I}_h u - u_h| + \delta_2 \max_\tau |\mathrm{I}_h u - u_h|^2$. Let $\theta = u_h - \mathrm{I}_h u \in S_{0h}$ and take $\varphi_h = \theta$. An application of Lemmas 4.1, 3.2 and 3.6, and the Hölder inequality yields

$$\alpha\|\theta\|_1^2 \le Ch\|\theta\|_1 + C(h\|\theta\|_{0,\infty} + \|\theta\|_{0,\infty}^2)\|\theta\|_{0,1}.$$

Recalling for Bramble [4] that

$$\|\theta\|_{0,\infty} \le C|\ln h|^{1/2}\|\nabla\theta\| \le C|\ln h|^{1/2}\|\theta\|_1 \tag{4.9}$$

holds for $\theta \in S_{0h}$ and by the well known Sobolev inequality

$$\|v\|_{0,p} \le C\|v\|_1, \qquad 1 \le p < \infty, \tag{4.10}$$

we get

$$\alpha\|\theta\|_1^2 \le Ch\|\theta\|_1 + C(h|\ln h|^{1/2}\|\theta\|_1 + |\ln h|\|\theta\|_1^2)\|\theta\|_1.$$

Omitting the common factor $\|\theta\|_1$, gives

$$\alpha\|\theta\|_1 \le Ch + Ch|\ln h|^{1/2}\|\theta\|_1 + C|\ln h|\|\theta\|_1^2. \tag{4.11}$$

For $h \le h'$, omitting the second term of the right-side implies

$$\|\theta\|_1 \le C_1 h + C_2|\ln h|\|\theta\|_1^2. \tag{4.12}$$

Now adopting a continuity argument by imitating the method by Frehse-Rannacher [24], yields

$$\|\theta\|_1 \le \|\mathrm{I}_h u - u_h\|_1 \le 2C_1 h. \tag{4.13}$$

For $s \in [0,1]$ considering the auxiliary semilinear elliptic problems (P^s): Find u^s such that

$$\begin{cases} -\frac{\partial}{\partial x}\Big(a_{11}\frac{\partial u^s}{\partial x} + a_{12}\frac{\partial u^s}{\partial y}\Big) - \frac{\partial}{\partial y}\Big(a_{21}\frac{\partial u^s}{\partial x} + a_{22}\frac{\partial u^s}{\partial y}\Big) + sf(u^s) = sg, \quad \text{in } \Omega, \\ u^s = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Obviously, for $s = 1$ this is our original problem (2.1) and for $s = 0$ we have $u^0 \equiv 0$ on $\bar{\Omega}$. We shall assume the following condition on Ω. For any $s \in [0,1]$, there is a solution u^s of problem (P^s) and there is a constant Γ such that the set

$$N_\Gamma = \Big\{\omega | \omega \in H^2(\Omega) \cap H_0^1(\Omega), \max_{\Omega} |u - \omega| < \Gamma\Big\}$$

is some neighborhood of exact solution u in (2.1).

We approximate problem (P^s) by the discrete problems (P_h^s): Find $u_h^s \in S_{0h}$ such that

$$a(u_h^s, \mathrm{I}_h^* v_h) + s(\mathrm{I}_h f(u_h^s), \mathrm{I}_h^* v_h) = s(g, \mathrm{I}_h^* v_h), \quad \forall v_h \in S_{0h}. \tag{4.15}$$

We intend to show that (P_h^s) is solvable. For each h, we define the set $E_h \subset [0,1]$ by

$$E_h = \Big\{s \in [0,1] | (\mathrm{P}_h^s) \text{ has a solution } u_h^s \in N_\Gamma \\ \text{and there holds } \|\mathrm{I}_h u^s - u_h^s\|_1 \le 2C_1 h\Big\},$$

where C_1 is the constant appearing in (4.12). Below gives some observations:

(i) E_h is not empty. In fact, for $s = 0, u^s = 0$ and $u_h^s = 0$ are the solutions of continuous and the discrete problem, respectively.

(ii) E_h is open in $[0,1]$. In fact, if $s \in E_h$ then (P_h^s) is solvable and using the monotonicity condition, we obtain the solvability of (P_h^s) for all t in a neighborhood of s via the implicit function theorem. By the implicit function theorem u_h^t depends continuously on t. Thus properly shorten the neighborhood such that the strict inequality $\|\mathrm{I}_h u^s - u_h^s\|_1 < 2C_1 h$ and $u_h^s \in N_\Gamma$ is still valid and we have $t \in E_h$ for these t.

(iii) E_h is closed. Let $s(j) \in E_h$ and $s(j) \to s, j \to \infty$. Since $u_h^{s(j)} \in N_\Gamma$ there is a cluster point u_h^s which is the unique solution of (P_h^s) and satisfies $\|\mathrm{I}_h u^s - u_h^s\|_1 \le 2C_1 h$. Recalling for (4.12) we conclude

$$\|\mathrm{I}_h u^s - u_h^s\|_1 \le C_1 h + 4C_2 C_1^2 |\ln h| h^2 \le C_1(1 + 4C_1 C_2 |\ln h| h) h,$$

then for $h \le h'' = h''(C_1, C_2)$, we have $4C_1 C_2 |\ln h| h < 1$ and $\|\mathrm{I}_h u^s - u_h^s\|_1 < 2C_1 h$, i.e. the strict inequality.

From (i)–(iii), we know that for $h \le \min(h', h'')$ the set E_h is not empty, closed and open with respect to $[0,1]$ and thus must coincide with $[0,1]$. Note that for $s = 1$, (P_h^1) is solvable. We prove that inequality (4.13) and $u_h \in N_\Gamma$ hold for appropriately small h.

Finally, the desired estimate (4.6) follows from (4.13) and the interpolation property

$$\|u - \mathrm{I}_h u\|_1 \le Ch\|u\|_2.$$

This completes the proof of this theorem. □

For the proof of the L^2-norm estimate, we shall employ a duality argument as the one used in [7, 21], Let us consider the another auxiliary problem. Let $\varphi \in H_0^1$ be such that

$$a(\varphi, v) + (f'(u)\varphi, v) = (u - u_h, v), \quad \forall v \in H_0^1. \tag{4.16}$$

Then the solution of (4.16) satisfies the following elliptic regularity estimate

$$\|\varphi\|_2 \le C\|u-u_h\|. \tag{4.17}$$

Theorem 4.2. *Assume $f'(s) > 0, f \in C^2(R), g \in H^1(\Omega)$. Let $u \in H_0^1(\Omega)\cap H^2(\Omega)$ be the solution of (2.1) and $u_h \in S_{0h}$ be the approximate solution of finite volume element method (2.5) with interpolated coefficients, respectively. Assume $\mathcal{J}_h$ is quasi-uniform triangular partition of domain Ω. Then*

$$\|u-u_h\| \le C(u,f,g)h^2, \tag{4.18}$$

holds for sufficiently small h.

Proof. First, in view of (4.16), we have

$$\begin{aligned}\|u-u_h\|^2 &= a(u-u_h,\varphi) + (f'(u)(u-u_h),\varphi)\\ &= \Big(a(u-u_h,\varphi-\mathrm{I}_h\varphi) + (f'(u)(u-u_h),\varphi-\mathrm{I}_h\varphi)\Big)\\ &\quad + \Big(a(u-u_h,\mathrm{I}_h\varphi) + (f'(u)(u-u_h),\mathrm{I}_h\varphi)\Big) =: \mathrm{I}_1+\mathrm{I}_2.\end{aligned} \tag{4.19}$$

Using the interpolation property, we can get

$$|\mathrm{I}_1| \le C(u,f)h\|u-u_h\|_1\|\varphi\|_2. \tag{4.20}$$

Notice (4.8) and rewrite I_2 as

$$\begin{aligned}\mathrm{I}_2 &= a(u,\mathrm{I}_h\varphi) - a(u_h,\mathrm{I}_h\varphi) + (f'(u)(u-u_h),\mathrm{I}_h\varphi)\\ &\quad - (g,\mathrm{I}_h^*\varphi) + a(u_h,\mathrm{I}_h^*\varphi) + (\mathrm{I}_hf(u_h),\mathrm{I}_h^*\varphi)\\ &= [(g,\mathrm{I}_h\varphi)-(g,\mathrm{I}_h^*\varphi)] - [a(u_h,\mathrm{I}_h\varphi)-a(u_h,\mathrm{I}_h^*\varphi)] - (f(u)-\mathrm{I}_hf(u),\mathrm{I}_h\varphi)\\ &\quad - [(\mathrm{I}_hf(u),\mathrm{I}_h\varphi-\mathrm{I}_h^*\varphi)] + (f'(u)(u-u_h),\mathrm{I}_h\varphi) - (\mathrm{I}_h(f(u)-f(u_h)),\mathrm{I}_h^*\varphi)\\ &= [(g,\mathrm{I}_h\varphi)-(g,\mathrm{I}_h^*\varphi) - [a(u_h,\mathrm{I}_h\varphi)-a(u_h,\mathrm{I}_h\varphi)] - (f(u)-\mathrm{I}_hf(u),\mathrm{I}_h\varphi)\\ &\quad - [(\mathrm{I}_hf(u),\mathrm{I}_h\varphi-\mathrm{I}_h^*\varphi)] + (f'(u)R,\mathrm{I}_h\varphi) - [(f'(u)\theta,\mathrm{I}_h\varphi)-(f'(u)\theta,\mathrm{I}_h^*\varphi)] + \sum_{\tau\in\mathcal{J}_h}(r,\mathrm{I}_h^*\varphi).\end{aligned}$$

Applying Lemmas 3.5, 3.7 and 3.9, and (4.9)–(4.10), we get

$$|\mathrm{I}_2| \le C\Big(h^2 + h\|u-u_h\|_1 + h|\ln h|^{1/2}\|\theta\|_1 + h|\ln h|\|\theta\|_1^2\Big)\|\varphi\|_1. \tag{4.21}$$

Therefore, substituting (4.20), (4.21) and (4.17) into (4.19) yields

$$\|u-u_h\|^2 \le |\mathrm{I}_1|+|\mathrm{I}_2| \le C\Big(h^2 + h\|u-u_h\|_1 + h|\ln h|^{1/2}\|\theta\|_1 + h|\ln h|\|\theta\|_1^2\Big)\|u-u_h\|.$$

Omitting the common factor $\|u-u_h\|$ gives

$$\|u-u_h\| \le C\Big(h^2 + h\|u-u_h\|_1 + h|\ln h|^{1/2}\|\theta\|_1 + h|\ln h|\|\theta\|_1^2\Big).$$

This, together with (4.6) and (4.13) in Theorem 4.1, gives the desired estimate (4.18). □

Theorem 4.3. *Assume $f'(s) > 0, f \in C^2(R), g \in H^1(\Omega)$. Let $u \in H_0^1(\Omega)\cap W^{2,\infty}(\Omega)$ be the solution of (2.1) and $u_h \in S_{0h}$ be the approximate solution of finite volume element method (2.5) with interpolated coefficients, respectively. Assume that the coefficients a_{12}, a_{21} in (2.1) satisfy $a_{12} = a_{21}$ and $\mathcal{J}_h$ is quasi-uniform triangular partition of domain Ω. Then*

$$\|u-u_h\|_{0,\infty} \le Ch^2|\ln h|, \tag{4.22}$$

where the constant C is dependent of u, f, g and independent of h.

Proof. By using the triangle inequality, we have

$$\|u-u_h\|_{0,\infty} \le \|u-\tilde{u}_h\|_{0,\infty} + \|\tilde{u}_h-u_h\|_{0,\infty},$$

where $\tilde{u}_h$ is the finite element approximation of u satisfying

$$a(\tilde{u}_h, v_h) + (f(\tilde{u}_h), v_h) = (g, v_h), \quad \forall v_h \in S_{0h}. \tag{4.23}$$

It has been shown in [7, 9, 31] that

$$\|u-\tilde{u}_h\|_{0,\infty} \le C(u,f)h^2|\ln h|, \tag{4.24}$$

$$\|u-\tilde{u}_h\|_1 \le C(u,f)h, \tag{4.25}$$

$$\|\tilde{u}_h\|_1 \le C. \tag{4.26}$$

Next, we turn our attention to the estimate of $\|\tilde{u}_h-u_h\|_{0,\infty}$. Let $P^* \in K_0 \subset \mathcal{J}_h$ such that $\|\tilde{u}_h-u_h\|_{0,\infty} = |(\tilde{u}_h-u_h)(P^*)|$ and $\delta_{P^*} \in C_0^\infty(\Omega)$ is a regularized Dirac δ-function satisfying

$$(\delta, v_h) = v_h(P^*).$$

Consider the corresponding regularized Green's function $G \in H_0^1(\Omega)$, defined by

$$a(G, v) + (f'(\tilde{u}_h)G, v) = (\delta_{P^*}, v), \quad \forall v \in H_0^1(\Omega). \tag{4.27}$$

Let $G_h \in S_{0h}$ be the finite element approximation of G, i.e.

$$a(G-G_h, v_h) + (f'(\tilde{u}_h)(G-G_h), v_h) = 0, \quad \forall v_h \in S_{0h}.$$

Then, in terms of (3.2) and (4.23), we can get

$$\begin{aligned}
\|\tilde{u}_h-u_h\|_{0,\infty} &= (\delta_{P^*}, \tilde{u}_h-u_h) = a(\tilde{u}_h-u_h, G_h) + (f'(\tilde{u}_h)(\tilde{u}_h-u_h), G_h) \\
&= (g, G_h) - (f(\tilde{u}_h), G_h) - a(u_h, G_h) + (f'(\tilde{u}_h)(\tilde{u}_h-u_h), G_h) \\
&\quad\ + a(u_h, \mathrm{I}_h^* G) + (\mathrm{I}_h f(u_h), \mathrm{I}_h^* G_h) - (g, \mathrm{I}_h^* G_h) \\
&= \big\{(g, G_h - \mathrm{I}_h^* G_h) - a(u_h, G_h - \mathrm{I}_h^* G_h)\big\} + \big\{(\mathrm{I}_h f(u_h), \mathrm{I}_h^* G_h) \\
&\quad\ - (f(u_h), G_h)\big\} + (f'(\tilde{u}_h)(\tilde{u}_h-u_h) - f(\tilde{u}_h) + f(u_h), G_h) \\
&=: \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5.
\end{aligned} \tag{4.28}$$

Using Lemma 3.7, Lemma 3.9 and Theorem 4.1, gives

$$\begin{aligned}
|\mathrm{I}_3| &\le Ch^2\|g\|_1\|G_h\|_1 + C(h\|u-u_h\|_1 + h^2\|u\|_2)\|G_h\|_1 \\
&\le C(u,g)h^2\|G_h\|_1.
\end{aligned} \tag{4.29}$$

Using Lemma 3.7 and the interpolation property, we have

$$\begin{aligned}
|\mathrm{I}_4| &= |(f(u_h), G_h - \mathrm{I}_h^* G_h)| + |(f(u_h) - \mathrm{I}_h f(u_h), \mathrm{I}_h^* G_h)| \\
&\le C(u,f)h^2\|G_h\|_1.
\end{aligned} \tag{4.30}$$

Using

$$\begin{aligned}
&f(u_h) - f(\tilde{u}_h) - f'(\tilde{u}_h)(u_h-\tilde{u}_h) \\
&= (u_h-\tilde{u}_h)^2 \int_0^1 f''(u_h - t(u_h-\tilde{u}_h))(t-1)\mathrm{d}t,
\end{aligned}$$

and (4.25) and Theorem 4.1, we get

$$\begin{aligned}|\mathrm{I}_5| &\le |(f'(\tilde{u}_h)(\tilde{u}_h - u_h) - f(\tilde{u}_h) + f(u_h), G_h)| \\ &\le C\|(\tilde{u}_h - u_h)^2\|\|G_h\| \le C_1 h^2 \|G_h\|. \end{aligned} \tag{4.31}$$

In addition, it follows from [7,32] that

$$\|G_h\|_1 \le C|\ln h|^{1/2}. \tag{4.32}$$

Combining (4.29)-(4.31) we obtain

$$\|\tilde{u}_h - u_h\|_{0,\infty} \le Ch^2|\ln h|^{1/2}.$$

From this and (4.24) we get

$$\|u - u_h\|_{0,\infty} \le C(1 + |\ln h|^{-1/2})h^2|\ln h|,$$

which gives the desired estimate (4.22) for sufficiently small h. □

5. Numerical Example

In this section we present a numerical experiment to verify the theoretical results. We consider the following semilinear elliptic problem

$$-\Delta u + u^3 = g, \quad \text{in } \Omega = (0,1)\times(0,1), \qquad u = 0, \quad \text{on } \partial\Omega, \tag{5.1}$$

where the function g is chosen, such that the known solution is

$$u(x,y) = y(1-x)\sin(x(1-y)).$$

Place a right triangular decomposition on the domain $\Omega = (0,1)\times(0,1)$ with the right-angle-side length $h = \frac{1}{N}$, $x_i = \frac{i}{N}$, $y_j = \frac{j}{N}$, $i,j = 0,1,\ldots,N$, see Fig. 5.1.

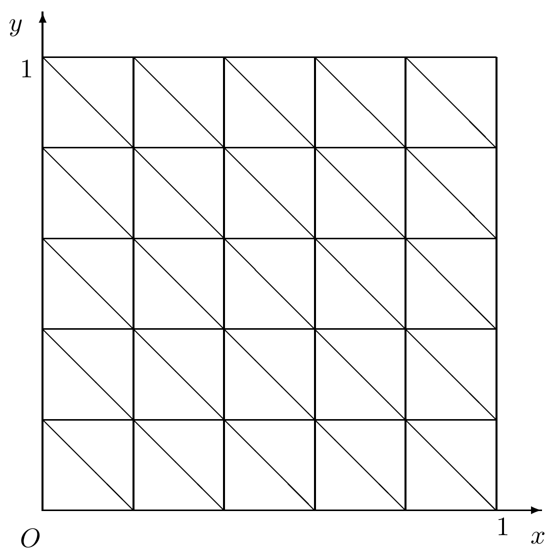

Fig. 5.1. The right triangulation of $\Omega = (0,1)\times(0,1)$ with the right-angle-side length $h = \frac{1}{5}$.

By using the linear triangular finite volume element method with interpolated coefficients, we obtain the numerical results as listed in Table 5.1. From Table 5.1, one can see that the triangular linear finite volume element with interpolated coefficients satisfies the results in our theoretical analysis.

Table 5.1. Errors of FVEM with interpolated coefficients.

	H^1-seminorm		L^2-norm		L^∞-norm	
h	Error	Rate	Error	Rate	Error	Rate
0.200	$8.4484e-4$		$5.1397e-4$		$6.0746e-4$	
0.100	$3.1565e-4$	1.4204	$1.2708e-4$	2.0159	$1.2431e-4$	2.2888
0.050	$9.0075e-5$	1.8091	$3.1677e-5$	2.0042	$2.8110e-5$	2.1448
0.025	$2.3786e-5$	1.9210	$7.9135e-6$	2.0011	$6.6846e-6$	2.0722

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