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ON L^2 ERROR ESTIMATE FOR WEAK GALERKIN FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS*

Fuzheng Gao
School of Mathematics, Shandong University, Jinan, Shandong 250100, China
Email: fzgao@sdu.edu.cn
Lin Mu
Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
Email: linmu@math.msu.edu

Abstract

A weak Galerkin finite element method with stabilization term, which is symmetric, positive definite and parameter free, was proposed to solve parabolic equations by using weakly defined gradient operators over discontinuous functions. In this paper, we derive the optimal order error estimate in L^2 norm based on dual argument. Numerical experiment is conducted to confirm the theoretical results.

Mathematics subject classification: 65M15, 65M60.
Key words: WG-FEMs, discrete weak gradient, parabolic problem, error estimate.

1. Introduction

We consider in this paper the approximation of a parabolic problem on a bounded domain $\Omega \subset R^2$ of the form

$$u_t - \nabla \cdot (a\nabla u) = f, \quad x \in \Omega, \quad 0 < t \le T, \tag{1.1a}$$
$$u = u^0, \qquad x \in \Omega, \quad t = 0, \tag{1.1b}$$

with homogenous Dirichlet boundary condition, where u_t is the time partial derivative of $u(x,t)$; $a(x)$ is an uniformly positive on $\bar{\Omega}$ and $a(x)$, $f(x,t)$ and $u^0(x)$ are assumed to be sufficiently smooth. Since the 1950s, scientists have formulated time-stepping procedures to numerically approximate the solutions of such problems.

Numerical methods for such parabolic problems can be classified as two categories. The first category consists of finite difference methods that use difference quotient to replace differential quotient and the other refers to as finite element methods, see, e.g., [3, 5, 6, 11–13, 17] and references in.

The WG-FEMs, which was first introduced by Wang and Ye [15] for solving the second order elliptic problems, are newly developed FEMs. The novel idea of WG-FEMs is to introduce weak functions and weak derivatives, and allows the use of totally discontinuous piecewise polynomials in the finite element procedure. Later, The WG -FEMs were studied from implementation point of view in [7] and applied to solve the Helmholtz problem with high wave numbers in [9].

A WG-FEM was introduced and analyzed for parabolic equations based on a discrete weak gradient arising from local RT [10]. Due to the use of RT elements, the WG finite element

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formulation of [4] was limited to classical finite element partitions of triangles ($d = 2$) or tetrahedra ($d = 3$). In our previous work, we presented a WG-FEM with stabilization term for a parabolic equation. This method is symmetric, positive definite and parameter free, and allows the use of partitions with arbitrary polygons in two dimensions, or polyhedra in three dimensions with certain shape regularity. Optimal convergence rate in H^1 norm and suboptimal convergence rate in L^2 norm for the WG approximation are derived. The objective of this paper is to derive an optimal order error estimate in L^2 norm based on dual argument technique for the solution of the WG-FEM.

The paper is organized as follows. Section 1 is introduction. In Section 2, we define weak gradient and present semi-discrete and full-discrete WG-FEM for problem (1.1). In Section 3, we establish the optimal order error estimates in L^2-norm to the WG-FEM for the parabolic problem based on dual argument. Finally, in Section 4 we give some numerical examples to verify the theory.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted as those in [1, 2].

2. A Weak Galerkin Finite Element Method

The variational form to (1.1) is seeking $u = u(x,t) \in L^2(0,T;H_0^1(\Omega))$, such that

$$(u_t, v) + a(u,v) = (f,v), \qquad \forall\, v \in H_0^1(\Omega), \quad t > 0, \tag{2.1a}$$

$$u(x,0) = u^0(x), \qquad x \in \Omega, \tag{2.1b}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$ and $a(\cdot,\cdot)$ is defined in (2.2).

$$a(v,w) = \int_\Omega a\nabla v \cdot \nabla w \mathrm{d}x. \tag{2.2}$$

It is well known that the solution to (2.1) is called generalized solution of (1.1).

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions [14]. Define $(u,v)_T = \int_T uv\mathrm{d}x$ and $\langle u,v\rangle_{\partial T} = \int_{\partial T} uv\mathrm{d}s$. We introduce a trial function space V_h, which is called weak Galerkin finite element space, as follows

$$V_h := \Big\{ v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_k(e), e \subset \partial T, \forall T \in \mathcal{T}_h \Big\}, \tag{2.3}$$

where T^0 and ∂T denote the interior and boundary of element $T \in \mathcal{T}_h$ respectively. Let $P_k(T^0)$ and $P_k(\partial T)$ be the sets of polynomials on T^0 and ∂T with degree no more than k respectively. v_0 represents the value of v on T^0 and v_b represents that of v on ∂T, respectively. We define V_h^0 as a subspace of V_h with zero boundary value, i.e.,

$$V_h^0 := \Big\{ v = \{v_0, v_b\} \in V_h, v_b \mid_{\partial T \bigcap \partial\Omega} = 0, \forall T \in \mathcal{T}_h \Big\}. \tag{2.4}$$

For each $v = \{v_0, v_b\} \in V_h$, we define the weak discrete gradient $\nabla_w v \in [P_{k-1}(T)]^2$ of v on each element T by the equation as:

$$(\nabla_w v, q)_T = -(v_0, \nabla \cdot q) + \langle v_b, q \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall q \in [P_{k-1}(T)]^2. \tag{2.5}$$

The semi-discrete WG-FE scheme for (1.1) as follows: Find $u_h = \{u_0(\cdot,t), u_b(\cdot,t)\} \in V_h^0$ for $(0 \le t \le T)$, such that

$$(u_{h,t}, v) + a_w(u_h, v) = (f, v_0), \qquad \forall\, v = \{v_0, v_b\} \in V_h^0, \tag{2.6a}$$

$$u_h(x,0) = Q_h u^0(x), \qquad x \in \Omega, \tag{2.6b}$$

where the bilinear form $a_w(v,w)$ is defined as

$$a_w(v,w) := \sum_{T\in\mathcal{T}_h} (a\nabla_w v, \nabla_w w)_T + \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle v_0 - v_b, w_0 - w_b\rangle_{\partial T}. \tag{2.7}$$

Here $Q_h u = \{Q_0 u, Q_b u\}$ in which Q_0 is the L^2 projection from $L^2(T)$ to $P_k(T)$ and Q_b is the L^2 projection from $L^2(e)$ to $P_k(e)$.

Now, we introduce a norm $|||\cdot|||_{w,1}$ as

$$|||v|||_{w,1} := \sqrt{\sum_{T\in\mathcal{T}_h} (\|\nabla_w v\|_{0,T}^2 + h_T^{-1}\|v_0 - v_b\|_{0,\partial T}^2)}, \tag{2.8}$$

which is a H^1-equivalent norm for conventional finite element functions with zero boundary value.

Let us now return to our semi-discrete problem in the formulation (2.6). A basic stability inequality for this problem (1.1) is as follows:

Theorem 2.1. *For the numerical solution to scheme* (2.6) *with initial setting* (2.6*b*)*, there is a good stability as follows*

$$\|u_h(t)\|_0^2 \le C\Big(\|u_h(0)\|_0^2 + \textstyle\int_0^t \|f(\tau)\|_0^2 d\tau\Big), \tag{2.9}$$

i.e., the numerical solution is stable with respect to initial approximate value and source term.

Proof. Taking $v = u_h$ in (2.6a), we get

$$(u_{h,t}(t), u_h(t)) + a_w(u_h(t), u_h(t)) = (f, u_h(t)).$$

From the definition of bilinear form $a_w(\cdot,\cdot)$, see equation (2.7), we know that

$$a_w(u_h(t), u_h(t)) \ge 0.$$

Based on this fact, it is easy to know that

$$(u_{h,t}(t), u_h(t)) \le (f, u_h(t)).$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h^2(t)\mathrm{d}x &= (u_{h,t}(t), u_h(t)) \le \int_\Omega f u_h(t)\mathrm{d}x \\ &\le C\Big(\int_\Omega f^2 \mathrm{d}x + \int_\Omega u_h^2(t)\mathrm{d}x\Big).\end{aligned}$$

Integrate the above inequality with respect to t, we have

$$\|u_h(t)\|_0^2 \le \|u_h(0)\|_0^2 + C\int_0^t \|f(\tau)\|_0^2 \mathrm{d}\tau + C\int_0^t \|u_h(\tau)\|_0^2 \mathrm{d}\tau.$$

Using Gronwall lemma, we complete the proof of this theorem 2.1. □

Let τ denote the time step size, and $t_n = n\tau$ $(n = 0, 1, \cdots)$, $u_h^n := u_h(t_n) = \{u_0^n, u_b^n\}$. At time $t = t_n$, adopting the backward Euler difference quotient

$$\bar{\partial}_t u_h^n = (u_h^n - u_h^{n-1})/\tau$$

to approximate $u_{h,t}$ in scheme (2.6), we get the fully-discrete WG-FE scheme: Find $u_h^n = \{u_0^n, u_b^n\} \in V_h^0$ for $n = 1, 2, \cdots$, such that

$$(\bar{\partial}_t u_h^n, v) + a_w(u_h^n, v) = (f^n, v), \quad \forall v \in V_h^0, \tag{2.10a}$$

$$u_h^0 = Q_h u^0(x), \tag{2.10b}$$

or we can equivalently write it as

$$(u_h^n, v) + \tau a_w(u_h^n, v) = (u_h^{n-1} + \tau f^n, v), \quad \forall v \in V_h^0, \tag{2.11a}$$

$$u_h^0 = Q_h u^0(x). \tag{2.11b}$$

The existence and uniqueness of its solution $u_h^n = \{u_0^n, u_b^n\}$ to (2.10) or (2.11) for a given $u_h^{n-1} = \{u_0^{n-1}, u_b^{n-1}\}$ can be readily proved.

3. Error Estimate

In this section, we will present optimal order priori error estimates in L^2-norm for the semi-discrete scheme (2.6) and fully-discrete scheme (2.10) or (2.11) for smooth solution of (1.1).

For simplicity of analysis, we assume that diffusion coefficient a is a piecewise constant with respect to the finite element partition $\mathcal{T}_h$. The corresponding results can be extended to variable coefficient case, provided that the coefficient function a is piecewise sufficiently smooth.

Below we denote C (maybe with indicates) as a positive constant solely depending on the exact solution, which may have different values in each occurrence.

3.1. Preliminaries

3.1.1. Notations of Sobolev space

Let Ω be any domain in R^2. In this paper, we still use the standard definition for the Sobolev space $W^{s,r}(\Omega)$, which consists of functions with (distributional) derivatives of order less than or equal to s in $L^r(\Omega)$ for $1 \le r \le +\infty$ and integer s. And their associated inner product $(\cdot, \cdot)_{s,r,\Omega}$, norm $\|\cdot\|_{s,r,\Omega}$, and seminorm $|\cdot|_{s,r,\Omega}$. Further, $\|\cdot\|_{\infty,\Omega}$ represents the norm on $L^\infty(\Omega)$, and $\|\cdot\|_{L^\infty([0,T];W^{s,r}(\Omega))}$ the norm on $L^\infty([0,T]; W^{s,r}(\Omega))$. See Adams [1] for more details.

3.1.2. Properties of finite element space

In our analysis, we shall adopt two kinds of finite element space associated with each element $T \in \mathcal{T}_h$. One is a scalar polynomial space in which the polynomial degree is no more than k on T^0 and ∂T, and the other is a vector valued polynomial space $[P_{k-1}(T)]^2$ which is used to define the discrete weak gradient ∇_w in (2.5). For convenience, we denote $[P_{k-1}(T)]^2$ by $G_{k-1}(T)$ which is called a local discrete gradient space.

In addition, we define two local L^2 projections in this paper. One is $Q_hv := \{Q_0v, Q_bv\}$ introduced after (2.7). The other is $\mathcal{Q}_h\mathbf{w}(x)$, which is defined as

$$\int_T \mathcal{Q}_h\mathbf{w}(x)\cdot q(x)\mathrm{d}x = \int_T \mathbf{w}(x)\cdot q(x)\mathrm{d}x, \quad \forall q(x)\in G_{k-1}(T). \tag{3.1}$$

The following three lemmas can be found in [8,15].

Lemma 3.1. *Let $\mathcal{Q}_h$ be the projection operator defined as in* (3.1). *Then, on each element $T\in\mathcal{T}_h$, we have the following relation*

$$\nabla_w(Q_h\phi) = \mathcal{Q}_h(\nabla\phi), \quad \forall\phi\in H^1(\Omega). \tag{3.2}$$

Lemma 3.2. *Let $\mathcal{T}_h$ be a finite element partition of domain Ω satisfying corresponding shape regularly assumptions. Then, for any $\phi\in H^{k+1}(\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h}\|\phi-Q_0\phi\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(\phi-Q_0\phi)\|_T^2 \le Ch^{2(k+1)}\|\phi\|_{k+1}^2, \tag{3.3}$$

$$\sum_{T\in\mathcal{T}_h}\|a(\nabla\phi-\mathcal{Q}_h(\nabla\phi))\|_T^2 \le Ch^{2k}\|\phi\|_{k+1}^2. \tag{3.4}$$

Lemma 3.3. *Assume that $\mathcal{T}_h$ is shape regular, we have*

$$\left|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0w-Q_bw, v_0-v_b\rangle_{\partial T}\right| \le Ch^k\|w\|_{k+1}|||v|||_{w,1}, \tag{3.5}$$

$$\left|\sum_{T\in\mathcal{T}_h}\langle a(\nabla w-\mathcal{Q}_h\nabla w)\cdot\mathbf{n}, v_0-v_b\rangle_{\partial T}\right| \le Ch^k\|w\|_{k+1}|||v|||_{w,1}, \tag{3.6}$$

for $\forall w\in H^{k+1}(\Omega)$ and $v=\{v_0,v_b\}\in V_h^0$.

3.2. Error estimate for semi-discrete WG scheme

To begin with, we will analyze semi-discrete WG scheme (2.6). To get an optimal order of error estimate in L^2, similar to Wheeler's projection as in [16], we define a projection E_hu onto V_h^0 for the exact solution $u\in H_0^1(\Omega)\bigcap H^2(\Omega)$ of problem (1.1) as follows:

$$a_w(E_hu,\chi) = (-\nabla\cdot(a\nabla u),\chi), \quad \forall\chi\in V_h^0. \tag{3.7}$$

Remark. The projection operator E_h has been used in [4] for analyzing WG without stabilization term for parabolic equations.

Note that E_hu is the standard WG-FEM solution applied to the second order elliptic equation if u is sufficiently smooth. We can derive the following important lemma from the results of [8] directly.

Lemma 3.4. *Assume that the exact solution of the problem* (1.1) *is so regular that $u\in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$|||Q_hu-E_hu|||_{w,1} \le Ch^k\|u\|_{k+1}. \tag{3.8}$$

$$\|Q_0u-E_hu\| \le Ch^{k+1}\|u\|_{k+1}. \tag{3.9}$$

Based on the relation between Q_hu and E_hu, we have

Theorem 3.1. *Let $u(x,t)$ and $u_h(x,t)$ be the solutions to the problem* (1.1) *and the semi-discrete WG scheme* (2.6)*, respectively. Assume that the exact solution is so regular that u, $u_t \in H_0^1(\Omega)\bigcap H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$\|u-u_h\|_0^2 \le C\Big(\|u^0-u_h^0\|_0^2 + h^{2(k+1)}\big(\|u^0\|_{k+1}^2 + \int_0^t \|u_\tau\|_{k+1}^2 d\tau\big)\Big). \tag{3.10}$$

Proof. Write

$$\rho = u - Q_hu, \quad \eta = Q_hu - E_hu, \quad e = E_hu - u_h.$$

Thus, we have

$$u - u_h = \rho + \eta + e, \tag{3.11}$$

where Q_h is the local L^2-projection operator and E_h is defined in (3.7) and $e = \{e_0, e_b\} = \{Q_0u - u_0, Q_bu - u_b\}$. Hence we have

$$\|\rho\| \le Ch^{k+1}\|u\|_{k+1}, \qquad \|\rho_t\| \le Ch^{k+1}\|u_t\|_{k+1}. \tag{3.12}$$

Making use of Lemma 3.4 leads to

$$\|\eta\| \le Ch^{k+1}\|u\|_{k+1}, \qquad \|\eta_t\| \le Ch^{k+1}\|u_t\|_{k+1}. \tag{3.13}$$

According to the definition of projection E_hu, for $\forall v \in V_h^0$ we get

$$\begin{aligned}
(e_t, v) + a_w(e, v) &= (E_hu_t, v) + a_w(E_hu, v) - (u_{h,t}, v) - a_w(u_h, v) \\
&= (E_hu_t, v) + a_w(E_hu, v) - (f, v) \\
&= (E_hu_t, v) - (\nabla\cdot(a\nabla u), v) - (f, v) \\
&= (E_hu_t, v) - (Q_hu_t, v) + (Q_hu_t, v) - (u_t, v) \\
&= -(\eta_t, v) - (\rho_t, v).
\end{aligned} \tag{3.14}$$

Choosing the test function $v = e$ in (3.14) and we can have

$$(e_t, e) + a_w(e, e) = -(\eta_t, e) - (\rho_t, e). \tag{3.15}$$

It is easy to arrive at

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|_0^2 + C_0|||e|||_{w,1}^2 &\le (e_t, e) + a_w(e, e) = -(\eta_t, e) - (\rho_t, e) \\
&\le C(\|\eta_t\|_0^2 + \|\rho_t\|_0^2 + \|e\|_0^2).
\end{aligned} \tag{3.16}$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|_0^2 \le C\Big(\|\eta_t\|_0^2 + \|\rho_t\|_0^2 + \|e\|_0^2\Big). \tag{3.17}$$

Integrating (3.17) with respect to t, we can get the following inequality easily as follows

$$\|e(t)\|_0^2 \le \|e(0)\|_0^2 + C\left(\int_0^t \|\eta_\tau\|_0^2 \mathrm{d}\tau + \int_0^t \|\rho_\tau\|_0^2 \mathrm{d}\tau + \int_0^t \|e\|_0^2 \mathrm{d}\tau\right). \tag{3.18}$$

By virtue of Lemmas 3.2 and 3.4,

$$\begin{aligned}
\|e(0)\|_0 &= \|E_hu^0 - u_h^0\|_0 = \|E_hu^0 - Q_hu^0 + Q_hu^0 - u^0 + u^0 - u_h^0\|_0 \\
&\le \|E_hu^0 - Q_hu^0\|_0 + \|Q_hu^0 - u^0\|_0 + \|u^0 - u_h^0\|_0 \\
&\le Ch^{k+1}\|u^0\|_{k+1} + \|u^0 - u_h^0\|_0.
\end{aligned} \tag{3.19}$$

A combination of (3.11)-(3.13), (3.18), (3.19) with *Gronwall* lemma leads to (3.10). □

3.3. Error estimate for fully discrete WG scheme

Theorem 3.2. *Let u^n and $\{u_h^n\}$ be the solutions to the parabolic equation* (1.1) *and the fully discrete WG scheme* (2.10)*, respectively. Then*

$$\|u^n - u_h^n\|_0 \leq C\left\{\|u^0 - u_h^0\|_0 + \tau\int_0^{t_n}\|u_{tt}\|_0 dt + h^{k+1}(\|u^0\|_{k+1} + \int_0^{t_n}\|u_t\|_{k+1}dt)\right\}. \quad (3.20)$$

Proof. We still write

$$\begin{aligned} u^n - u_h^n &= u^n - Q_h u^n + Q_h u^n - E_h u^n + E_h u^n - u_h^n \\ &\equiv \rho^n + \eta^n + e^n, \end{aligned} \quad (3.21)$$

where $u^n = u(t_n)$ for convenience.

It follows from Lemmas 3.2 and 3.4 that

$$\|\rho^n\|_0 \leq Ch^{k+1}\|u^n\|_{k+1} \leq Ch^{k+1}\Big(\|u^0\|_{k+1} + \int_0^{t_n}\|u_\tau\|_{k+1}\mathrm{d}\tau\Big), \quad (3.22)$$

$$\|\eta^n\|_0 \leq Ch^{k+1}\|u^n\|_{k+1} \leq Ch^{k+1}\Big(\|u^0\|_{k+1} + \int_0^{t_n}\|u_\tau\|_{k+1}\mathrm{d}\tau\Big). \quad (3.23)$$

For e^n, we have

$$\begin{aligned} (\bar{\partial}_t e^n, v) + a_w(e^n, v) &= (\bar{\partial}_t E_h u^n, v) + a_w(E_h u^n, v) - (\bar{\partial}_t u_h^n, v) - a_w(u_h^n, v) \\ &= (\bar{\partial}_t E_h u^n, v) + a_w(E_h u^n, v) - (f^n, v) \\ &= (\bar{\partial}_t E_h u^n, v) - (\nabla\cdot(a\nabla u^n), v) - (f^n, v) \\ &= (\bar{\partial}_t E_h u^n, v) - (u_t(t_n), v) \\ &= -(\bar{\partial}_t \eta^n, v) - (\bar{\partial}_t \rho^n, v) - (u_t(t_n) - \bar{\partial}_t u^n, v), \end{aligned}$$

i.e.,

$$(\bar{\partial}_t e^n, v) + a_w(e^n, v) = -(\bar{\partial}_t \eta^n, v) - (\bar{\partial}_t \rho^n, v) - (u_t(t_n) - \bar{\partial}_t u^n, v). \quad (3.24)$$

Choosing $v = e^n$ in (3.24), we easily get

$$\begin{aligned} \|e^n\|_0 &\leq \|e^{n-1}\|_0 + C\tau\Big(\|\bar{\partial}_t\eta^n\|_0 + \|\bar{\partial}_t\rho^n\|_0 + \|u_t(t_n) - \bar{\partial}_t u^n\|_0\Big) \\ &\equiv \|e^{n-1}\|_0 + C\tau\Big(\mathfrak{R}_1^n + \mathfrak{R}_2^n + \mathfrak{R}_3^n\Big) \\ &\leq \|e^0\|_0 + C\tau\left(\sum_{i=1}^n \mathfrak{R}_1^i + \sum_{i=1}^n \mathfrak{R}_2^i + \sum_{i=1}^n \mathfrak{R}_3^i\right). \end{aligned} \quad (3.25)$$

For $\|e^0\|_0$, we have

$$\|e^0\|_0 \leq Ch^{k+1}\|u^0\|_{k+1} + \|u^0 - u_h^0\|_0. \quad (3.26)$$

Note that

$$\begin{aligned} \mathfrak{R}_1^i &= \|\bar{\partial}_t\eta^i\|_0 = \frac{1}{\tau}\left\|\int_{t_{i-1}}^{t_i}(Q_h - E_h)u_t\mathrm{d}t\right\|_0, \\ \mathfrak{R}^i &= \|\bar{\partial}_t\rho^i\|_0 = \frac{1}{\tau}\left\|\int_{t_{i-1}}^{t_i}(Q_h - I)u_t\mathrm{d}t\right\|_0, \\ \mathfrak{R}_3^i &= \|\bar{\partial}_t u(t_i) - u_t(t_i)\|_0 = \frac{1}{\tau}\left\|\int_{t_{i-1}}^{t_i}(t - t_{i-1})u_{tt}\mathrm{d}t\right\|_0. \end{aligned}$$

From Lemmas 3.4 and 3.2,

$$\sum_{i=1}^{n}\mathfrak{R}_1^i \le \frac{1}{\tau}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_j} Ch^{k+1}\|u_t\|_{k+1}\mathrm{d}t \le C\tau^{-1}h^{k+1}\int_0^{t_n}\|u_t\|_{k+1}\mathrm{d}t, \tag{3.27}$$

$$\sum_{i=1}^{n}\mathfrak{R}_2^i \le \frac{1}{\tau}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_j} Ch^{k+1}\|u_t\|_{k+1}\mathrm{d}t \le C\tau^{-1}h^{k+1}\int_0^{t_n}\|u_t\|_{k+1}\mathrm{d}t. \tag{3.28}$$

Similarly

$$\sum_{i=1}^{n}\mathfrak{R}_3^i \le \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|u_{tt}\|_0\mathrm{d}t = \int_0^{t_n}\|u_{tt}\|_0\mathrm{d}t. \tag{3.29}$$

A combination of (3.25)-(3.29) leads to (3.20). This completes the proof of Theorem 3.2. $\square$

4. Numerical Experiment

In this section, we present three numerical examples and consider the following parabolic problem with proper Dirichlet boundary condition and initial condition.

$$u_t - \mathrm{div}(\mathbf{D}\nabla u) = f, \quad \text{in} \quad \Omega\times J, \tag{4.1}$$

In all three numerical examples, for simplicity, we let $D = 1, 10$, Ω be a unit square, i.e., $\Omega = [0,1]\times[0,1]$, and time interval be $J = (0,T) = (0,1)$. One can determine the initial and boundary conditions and source term $f(x,t)$ according to the corresponding analytical solution of each example.

We construct triangular mesh as follows. Firstly, we partition the square domain $\Omega = (0,1)\times(0,1)$ into $N\times N$ sub-squares uniformly to obtain the square mesh. Then, we divide each square element into two triangles by the diagonal line with a negative slope so that we complete the constructing of triangular mesh. Let $h = 1/N(N = 4, 8, 16, 32, 64)$ be mesh sizes for triangular meshes. In the following three numerical examples, we choose the same time step $\tau = 1/100$.

In the first example, the analytical solution is

$$u = \sin(\pi x)\sin(\pi y)\exp(x+y+t). \tag{4.2}$$

For a set of simulations, different diffusion coefficients are taken, and their corresponding L^2-norm errors and convergence rates are listed in Table 4.1 for $D = 1$ and $D = 10$.

Table 4.1: Numerical results of the first example.

	$D=1$		$D=10$	
N	L_2 error	L_2 order	L_2 error	L_2 order
4	1.4332e-00		1.0095e+01	
8	3.6953e-01	1.96	2.5588e-00	1.98
16	9.3643e-02	1.98	6.4177e-01	2.00
32	2.4031e-02	1.96	1.6063e-01	2.00
64	6.6010e-03	1.86	4.0226e-02	2.00

In the second example, the analytical solution is

$$u = \sin(\pi x)\sin(\pi y)\exp(x+y-t). \tag{4.3}$$

Numerical error results and convergence rate are listed in Table 4.2 for $D = 1$ and $D = 10$ based on the same triangular mesh as in the first example.

Table 4.2: Numerical results of the second example.

	$D = 1$		$D = 10$	
N	L_2 error	L_2 order	L_2 error	L_2 order
4	2.0699e-01		1.4042e-00	
8	5.2769e-02	1.97	3.5062e-01	2.00
16	1.3340e-02	1.98	8.7607e-02	2.00
32	3.4267e-03	1.96	2.1906e-02	2.00
64	9.4697e-04	1.86	5.4847e-03	2.00

In the third example, the analytical solution is

$$u = x(1-x)y(1-y)\exp(-t). \tag{4.4}$$

Numerical error results and convergence rate are listed in Table 4.3 for $D = 1$ and $D = 10$ based on the same triangular mesh as that of the first example.

Table 4.3: Numerical results of the third example.

	$D = 1$		$D = 10$	
N	L_2 error	L_2 order	L_2 error	L_2 order
4	4.2506e-03		2.8688e-02	
8	1.0767e-03	1.98	7.1439e-03	2.00
16	2.7216e-04	1.98	1.7843e-03	2.00
32	7.0363e-05	1.95	4.4617e-04	2.00
64	1.9899e-05	1.82	1.1176e-04	2.00

All the three numerical examples given above are in good confirmly with the theoretical analysis in Section 3, which show that the WG finite element method (2.10) is stable and second order convergence in L_2 norm.

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