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ON BLOCK PRECONDITIONERS FOR PDE-CONSTRAINED OPTIMIZATION PROBLEMS*

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Abstract

Recently, Bai proposed a block-counter-diagonal and a block-counter-triangular preconditioning matrices to precondition the GMRES method for solving the structured system of linear equations arising from the Galerkin finite-element discretizations of the distributed control problems in (Computing 91 (2011) 379-395). He analyzed the spectral properties and derived explicit expressions of the eigenvalues and eigenvectors of the preconditioned matrices. By applying the special structures and properties of the eigenvector matrices of the preconditioned matrices, we derive upper bounds for the 2-norm condition numbers of the eigenvector matrices and give asymptotic convergence factors of the preconditioned GMRES methods with the block-counter-diagonal and the block-counter-triangular preconditioners. Experimental results show that the convergence analyses match well with the numerical results.

Mathematics subject classification: 65F08, 65F10.
Key words: PDE-constrained optimization, GMRES method, Preconditioner, Condition number, Asymptotic convergence factor.

1. Introduction

Preconditioning technique as an efficient tool has been widely applied in Krylov subspace methods for solving linear systems arising from discretizations of partial differential equations. In [3], Bai considered using the preconditioned Krylov subspace methods to solve the linear system emerging from the following distributed control problem

$$\min_{u,f} \frac{1}{2}\|u-u_*\|_2^2+\beta\|f\|_2^2, \tag{1.1}$$

$$\text{subject to} \quad -\nabla^2 u = f \quad \text{in } \Omega, \tag{1.2}$$

$$\text{with} \quad u=g \quad \text{on} \quad \partial\Omega_1 \quad \text{and} \quad \frac{\partial u}{\partial n}=g \quad \text{on } \partial\Omega_2, \tag{1.3}$$

where the domain $\Omega\subset\mathbb{R}^2$ or $\mathbb{R}^3$, $\partial\Omega_1$ and $\partial\Omega_2$ are distinct, $\partial\Omega_1\cup\partial\Omega_2=\partial\Omega$ and $\partial\Omega_1\cap\partial\Omega_2=\emptyset$, u_* is the known desired state. This problem was first introduced by Lions in [10]. We need to find u which satisfies the PDE problem (1.1)-(1.3) and is as close to u_* as possible in L_2-norm sense. A recent reference on this topic can be found in [9].

By adopting the discretize-then-optimize approach and employing the Galerkin finite element method in the discretization, the PDE-constrained optimization problem (1.1)-(1.3) can be transformed into a discrete analogue of the minimization problem. By applying the Lagrange

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multiplier technique to the minimization problem, we find that $\mathbf{f}$ and $\mathbf{u}$ can be defined by the following linear system

$$\boldsymbol{A}\mathbf{x} \equiv \begin{pmatrix} 2\beta\boldsymbol{M} & \mathbf{0} & -\boldsymbol{M} \\ \mathbf{0} & \boldsymbol{M} & \boldsymbol{K}^T \\ -\boldsymbol{M} & \boldsymbol{K} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix} \equiv \mathbf{g}, \tag{1.4}$$

where $\boldsymbol{M} \in \mathbb{R}^{m\times m}$ is the symmetric positive definite mass matrix, $\boldsymbol{K} \in \mathbb{R}^{m\times m}$ is the symmetric stiffness matrix (the discrete Laplacian), $\mathbf{d} \in \mathbb{R}^m$ contains the terms coming from the boundary values of the discrete solution, $\mathbf{b} \in \mathbb{R}^m$ is the Galerkin projection of the desired state u_* and λ is a vector of Lagrange multipliers, see also [7]. (1.4) is a saddle point problem if we write it in a 2-by-2 block form, see, for instance, [1, 2, 6]. Due to the finite element discretization, $\boldsymbol{M}$ and $\boldsymbol{K}$ are very large and sparse, the matrix $\boldsymbol{A}$ is large and sparse, too. By making use of the easiness of matrix-vector multiplications and linear computation in Krylov subspace methods, many preconditioned Krylov subspace methods have been proposed for solving (1.4), see, for instance, [3, 5, 8, 11–13]. Specifically, Bai applied the preconditioned GMRES method to solve the system (1.4) in [3]. He introduced two efficient preconditioners $\boldsymbol{P_{BCD}}$ and $\boldsymbol{P_{BCT}}$ to accelerate convergence rates of the GMRES method. $\boldsymbol{P_{BCD}}$ is a block-counter-diagonal preconditioner of form

$$\boldsymbol{P_{BCD}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -\boldsymbol{M} \\ \mathbf{0} & \boldsymbol{M} & \mathbf{0} \\ -\boldsymbol{M} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{1.5}$$

and $\boldsymbol{P_{BCT}}$ is a block-counter-triangular preconditioner of form

$$\boldsymbol{P_{BCT}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -\boldsymbol{M} \\ \mathbf{0} & \boldsymbol{M} & \boldsymbol{K}^T \\ -\boldsymbol{M} & \boldsymbol{K} & \mathbf{0} \end{pmatrix}. \tag{1.6}$$

It is clearly to see that the computation of $\boldsymbol{P_{BCD}}$ or $\boldsymbol{P_{BCT}}$ only requires to solve three linear sub-systems with the same coefficient matrix $\boldsymbol{M}$, and does not need to solve any linear sub-system with coefficient matrix $\boldsymbol{K}$. Therefore, the implementations of the preconditioned GMRES methods with these preconditioners for (1.4) are easy and effective.

In [3], the author also gave the spectral properties of the preconditioned matrices $\boldsymbol{P_{BCD}^{-1}A}$ and $\boldsymbol{P_{BCT}^{-1}A}$.

Theorem 1.1. (Theorem 2.1 in [3]) *Let $\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$ be the coefficient matrix of the saddle-point problem* (1.4) *and $\boldsymbol{P_{BCD}} \in \mathbb{R}^{3m\times 3m}$ be the block-counter-diagonal preconditioner of $\boldsymbol{A}$ defined in* (1.5). *Assume that v_l is an eigenvalue and $\mathbf{x}^{(l)} \in \mathbb{C}^m$ is the corresponding eigenvector of the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$, $l = 1, \ldots, m$, where $v_l > 0$ $(l = 1, \ldots, m)$. Then*

1. *the eigenvalues of the preconditioned matrix $\boldsymbol{P_{BCD}^{-1}A}$ are*

$$\lambda_k^{(l)} := 1 - \sqrt[3]{2\beta v_l}\, e^{\frac{(2k+1)\pi\imath}{3}}, \quad k = 0, 1, 2, \ l = 1, \ldots, m,$$

where $\imath$ denotes the imaginary unit;

2. *the eigenvectors of the preconditioned matrix $\boldsymbol{P_{BCD}^{-1}A}$ are*

$$\begin{pmatrix} \mathbf{x}^{(l)} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ -\boldsymbol{M}^{-1}\boldsymbol{K}^T\mathbf{x}^{(l)} \\ \mathbf{0} \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{x}^{(l)} \end{pmatrix}, \quad l = 1, \ldots, m.$$