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ON AUGMENTED LAGRANGIAN METHODS FOR SADDLE-POINT LINEAR SYSTEMS WITH SINGULAR OR SEMIDEFINITE $(1,1)$ BLOCKS*

Tatiana S. Martynova
Computing Center, Southern Federal University, Rostov-on-Don, Russia
Email: martynova@sfedu.ru

Abstract

An effective algorithm for solving large saddle-point linear systems, presented by Krukier et al., is applied to the constrained optimization problems. This method is a modification of skew-Hermitian triangular splitting iteration methods. We consider the saddle-point linear systems with singular or semidefinite $(1,1)$ blocks. Moreover, this method is applied to precondition the GMRES. Numerical results have confirmed the effectiveness of the method and showed that the new method can produce high-quality preconditioners for the Krylov subspace methods for solving large sparse saddle-point linear systems.

Mathematics subject classification: 65F10, 65F50.
Key words: Hermitian and skew-Hermitian splitting, Saddle-point linear system, Constrained optimization, Krylov subspace method.

1. Introduction

Large sparse linear systems of saddle-point type arise in different applications. In many cases the matrix of such linear system has zero $(2,2)$ block. Consider iterative solution of the large sparse indefinite system of linear equations in block-structured form

$$\begin{pmatrix} M & E^T \\ E & 0 \end{pmatrix} \begin{pmatrix} u \\ \mu \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

The matrix $M \in \mathbb{R}^{p\times p}$ is assumed to be symmetric positive semidefinite, the matrix $E \in \mathbb{R}^{q\times p}$ has full row rank, $q \le p$, $f \in \mathbb{R}^p$ and $g \in \mathbb{R}^q$ are two given vectors. Here E^T denotes the transpose of the matrix E. We assume that matrices M and E have no nontrivial null vectors. It is the condition that guarantees the existence and uniqueness of the solution; see [2].

This linear system corresponds to minimizing the quadratic objective functional $J(u) \equiv \frac{1}{2}u^T Mu - u^T f$, subject to q linear constraints $Eu = g$. The Lagrangian functional $\mathcal{L}(u,\mu) = J(u)+\mu^T(Eu-g)$ is associated with this constrained minimization problem, μ denotes the vector of Lagrange multipliers. Here M is the Hessian of the quadratic function to be minimized, and E is the Jacobian of the linear constraints [10, 11, 14].

A number of solvers have been developed for saddle-point linear systems in recent years, for example, projection methods [8], null space methods [1], HSS-like methods [3, 4, 6, 9], generalized successive overrelaxation methods [7], SSOR-like methods [18], and so on.

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We consider cases when the $(1,1)$ block in the linear system is semidefinite or singular. Frequently the singularity appears in the form of semidefiniteness. Our approach is based on augmentation of the $(1,1)$ block. We replace the $(1,1)$ block by a matrix that is much easier to invert. We employ the augmented Lagrangian method, the matrix M will be replaced by a positive definite matrix, and iterative methods can be applied to solve the augmented linear system. We replace the quadratic objective functional $J(u)$ by a regularized functional $J_\gamma(u) \equiv J(u)+\gamma\|Eu-g\|_W^2$, where $\gamma > 0$ is a parameter, $W = W^T > 0$ is a weighting matrix of size q, and $\|Eu-g\|_W^2 = (Eu-g)^T W(Eu-g)$. The augmented Lagrangian $\mathcal{L}_\gamma(u,\mu)$ associated with the minimization of $J_\gamma(u)$ is defined as [13]:

$$\mathcal{L}_\gamma(u,\mu) = \mathcal{L}(u,\mu) + \frac{\gamma}{2}\|Eu-g\|_W^2. \tag{1.1}$$

In the augmented saddle-point linear system, matrix M will be replaced by the matrix $\widetilde{M} \equiv M + \gamma E^T W E$ that will be positive definite for $\gamma > 0$. Applying the first derivative test to determine the saddle point of $\mathcal{L}_\gamma(u,\mu)$ yields the following linear system [13,14]:

$$\begin{pmatrix} \widetilde{M} & E^T \\ E & 0 \end{pmatrix} \begin{pmatrix} u \\ \mu \end{pmatrix} = \begin{pmatrix} f + \gamma E^T W g \\ g \end{pmatrix}. \tag{1.2}$$

Clearly, this linear system has precisely the same solution as the original one.

Recently in [15] the authors presented generalized skew-Hermitian triangular splitting iteration method (GSTS) for solving large non-Hermitian linear systems. The GSTS iteration method reduces to the skew-Hermitian triangular splitting (STS) iteration method studied in [17] and the product-type skew-Hermitian triangular splitting (PSTS) iteration method established in [16]. Then authors applied the GSTS iteration method to solve non-Hermitian saddle-point linear systems, proved its convergence under suitable restrictions on the iteration parameters, and implemented the method by solving the Stokes problem [15].

For iteratively solving linear systems arising in the constrained optimization problems, we use the GSOR [7] and the GSTS iteration methods. Numerical results show that the GSTS iteration method is effective for solving saddle-point linear systems arising in the constrained optimization problems with singular or semidefinite $(1,1)$ blocks by the augmented Lagrangian method.

2. Iteration Methods

We can rewrite the saddle-point linear system into an equivalent non-symmetric form [10,11]

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \tag{2.1a}$$

where

$$\mathcal{A} = \begin{pmatrix} M & E^T \\ -E & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} u \\ \mu \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} f \\ -g \end{pmatrix}. \tag{2.1b}$$

The matrix $\mathcal{A} \in \mathbb{R}^{(p+q)\times(p+q)}$ is positive stable now, that is, the eigenvalues of $\mathcal{A}$ have positive real parts [9,11]. Analogous to [5] the matrix $\mathcal{A}$ can be split into its symmetric and skew-symmetric parts as

$$\mathcal{A} = \mathcal{A}_{\mathcal{H}} + \mathcal{A}_{\mathcal{S}}, \tag{2.2}$$