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A PRIORI AND A POSTERIORI ERROR ESTIMATES OF A WEAKLY OVER-PENALIZED INTERIOR PENALTY METHOD FOR NON-SELF-ADJOINT AND INDEFINITE PROBLEMS*

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Abstract

In this paper, we study a weakly over-penalized interior penalty method for non-self-adjoint and indefinite problems. An optimal a priori error estimate in the energy norm is derived. In addition, we introduce a residual-based a posteriori error estimator, which is proved to be both reliable and efficient in the energy norm. Some numerical testes are presented to validate our theoretical analysis.

Mathematics subject classification: 65N15, 65N30.
Key words: Interior penalty method, Weakly over-penalization, Non-self-adjoint and indefinite, A priori error estimate, A posteriori error estimate.

1. Introduction

We are devoted to studying a weakly over-penalized interior penalty (WOPIP) method [7] for the following non-self-adjoint and indefinite problems

$$
\begin{aligned}
-\nabla\cdot(a\nabla u)+\mathbf{b}\cdot\nabla u+cu=f, &\quad \text{in} \quad \Omega,\\
u=0, &\quad \text{on} \quad \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset R^2$ is a bounded polygonal domain with boundary $\partial\Omega$. Here we assume that the data of (1.1), i.e., $\mathbf{D}=(a,\mathbf{b},c)$ satisfy the following property:

1. There exists $a_0>0$ such that $0<a_0<a$ and $c\geq 0$;
2. $a\in W^1_\infty(\Omega)$, $\mathbf{b}\in\big(L^\infty(\Omega)\big)^2$ and $c\in L^\infty(\Omega)$ with $M=\max\{\|a\|_{L^\infty(\Omega)},\|\mathbf{b}\|_{L^\infty(\Omega)},\|c\|_{L^\infty(\Omega)}\}$;
3. $f\in L^2(\Omega)$.

The WOPIP method belongs to a class of discontinuous Galerkin (DG) methods, which was first proposed in [7] by Brenner et al. to solve second order elliptic equations. DG methods

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for elliptic problems have been initially proposed in [2, 31] in the 1970s-1980s. In recent years they have gained much interest due to their suitability for hp-adaptive techniques, flexibility in handling inhomogeneous boundary conditions and curved boundaries, and their flexibility in handling highly nonuniform and unstructured meshes. The reader is referred to [14] for applications of these methods for a wide variety of problems, and to [3] for an over review of these methods for elliptic problems and their a priori error analysis. For more details of the a priori error estimates for second elliptic problems, please refer to [23]. For the theory of a posteriori error bounds for DG methods, the residual-based error estimators measured in mesh-dependent energy norms have been presented in [5, 19, 20, 22, 24], and further been studied in [1, 33]. Some other work on the a posteriori error estimates of DG methods can be found in [15, 26, 28, 29]. For the WOPIP method for second order equations, its a priori error estimate was provided in [7], where some advantages of this method were also discussed, e.g., compared with many well-known DG methods presented in [3], the WOPIP method has less computational complexity and is easy to implement. Subsequently, a residual-based posteriori error estimator was presented in [8]. More applications of the WOPIP methods are to use them to solve the biharmonic problem [9] and Stokes equations [4].

The non-self-adjoint and indefinite problems (1.1) often appear in dealing with flow in porous media. To the best of our knowledge, there exists no work on the a posteriori error estimates of DG methods for non-self-adjoint and indefinite problems. The main objective of this paper is to give a residual-based error estimator of the WOPIP DG method for (1.1). In this case, two main difficulties should be overcome, one arises from the effect of a nonsymmetric and indefinite bilinear form, the other stems from the nonconformity of the WOPIP DG method.

The rest of our paper is organized as follows. We introduce some notations and recall the WOPIP method in Section 2. An optimal a priori error estimate of the WOPIP method in the energy norm is provided in Section 3. A residual-based a posteriori error estimator of the WOPIP method is presented in Section 4. Moreover, both the upper bound and lower bound of the error estimator are proved in the energy norm. Finally, some numerical experiments which validate our theoretical results are given in Section 5.

2. Preliminaries and Notations

For a bounded domain $\mathcal{D}$ in R^2, we denote by $H^s(\mathcal{D})$ the standard Sobolev space of functions with regularity exponent $s \geq 0$, associated with norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. When $s = 0$, $H^0(\mathcal{D})$ can be written by $L^2(\mathcal{D})$. When $\mathcal{D} = \Omega$, the norm $\|\cdot\|_{s,\Omega}$ is simply written by $\|\cdot\|_s$. $H_0^s(\mathcal{D})$ is the subspace of $H^s(\mathcal{D})$ with vanishing trace on $\partial\mathcal{D}$.

Let $\mathcal{T}_h$ be a regular decompositions of Ω into triangles $\{T\}$, h_T denotes the diameter of T and $h = \max\limits_{T\in\mathcal{T}_h} h_T$. Denote ε_h^0 by the set of interior edges of elements in $\mathcal{T}_h$, and ε_h^∂ by the set of boundary edges. Set $\varepsilon_h = \varepsilon_h^0 \cup \varepsilon_h^\partial$. The length of any edge $e \in \varepsilon_h$ is denoted by h_e. Further, we associate a fixed unit normal $\mathbf{n}$ with each edge $e \in \varepsilon_h$ such that for edges on the boundary $\partial\Omega$, $\mathbf{n}$ is the exterior unit normal.

Let e be an interior edge in ε_h^0 shared by elements T_1 and T_2. For a scalar piecewise smooth function φ, with $\varphi^i = \varphi|_{T_i}$, we define the following jump by

$$[\![\varphi]\!] = \varphi^1 - \varphi^2, \quad \text{on} \;\; e \in \varepsilon_h^0.$$

For a boundary edge $e \in \varepsilon_h^\partial$, we set

$$[\![\varphi]\!] = \varphi.$$