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MULTIGRID METHOD FOR FLUID DYNAMIC PROBLEMS*

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Abstract

This paper covers the review and some aspects of using Multigrid method for fluid dynamics problems. The main development stages of multigrid technics are presented. Some approaches for solving Navier-Stokes equations and convection- diffusion problems are considered.

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Key words: Navier-Stokes equation, Convection-diffusion, Finite element method, Multigrid method.

1. Introduction

Fluid dynamics and transport phenomena, such as heat and mass transfer, play an important role in human life. Gases and liquids surround us, flow inside our bodies have a profound influence on the environment in which we live. Fluid flows produce winds, rains, floods, and hurricanes. Convection and diffusion are responsible for temperature fluctuations and transport of pollutants in air, water or soil.

The ability to understand, predict, and control transport phenomena is essential for many industrial applications, such as aerodynamic shape design, oil recovery from an underground reservoir, or multiphase/multicomponent flows in furnaces, heat exchangers, and chemical reactors. This ability offers substantial economic benefits and contributes to human well-being. Heating, air conditioning, and weather forecast have become an integral part of our everyday life. Most people take such things for granted and hardly ever think about the physics and mathematics behind them [1]. In physics fluid dynamics is a subdiscipline of fluid mechanics that deals with fluid flow-the natural science of fluids (liquids and gases) in motion. It has several subdisciplines itself, including aerodynamics (the study of air and other gases in motion) and hydrodynamics (the study of liquids in motion).

Before the twentieth century, hydrodynamics was synonymous with fluid dynamics. This is still reflected in names of some fluid dynamics topics, like magnetohydrodynamics and hydrodynamic stability, both of which can also be applied to gases.

Fluid dynamics has a wide range of applications, including calculating forces and moments on aircraft, determining the mass flow rate of petroleum through pipelines, predicting weather patterns, understanding nebulae in interstellar space. Some of its principles are even used in traffic engineering, where traffic is treated as a continuous fluid.

Computational fluid dynamics, usually abbreviated as CFD, is a branch of fluid mechanics that uses numerical methods and algorithms to solve and analyze problems that involve fluid flows. The fundamental basis of almost all CFD problems are the Navier-Stokes equations,

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which define any single-phase (gas or liquid, but not both) fluid flow. These equations can be simplified by removing terms describing viscous actions to yield the Euler equations. Further simplification, by removing terms describing vorticity yields the full potential equations. Finally, for small perturbations in subsonic and supersonic flows (not transonic or hypersonic) these equations can be linearized to yield the linearized potential equations.

The other most common equation in the computational fluid dynamics field is the convection-diffusion equation. Mathematical models that involve a combination of convective and diffusive processes are among the most widespread in all the sciences. Research of these processes is especially important and difficult when convection is dominant. At the same time, convection-diffusion equations are used as tests in researching iterative methods for solving systems of strongly nonsymmetric linear equations.

During the simulation of some physical phenomena, in the CFD, the solution of large linear systems is usually required. With the ongoing increase of the complexity of the problems to treat, the solution phase may be very costly. It is not sufficient to use the latest technology of computers. An effort should be put into algorithms for solving such systems. For such systems, involving several millions of degrees of freedoms, direct methods [2] are not convenient, and iterative methods [3] usually suffer from a low convergence speed. Hybrid methods, like domain decomposition methods [4], can be considered. These methods consist to split the global system to solve into multiple sub-systems, each subsystem being solved independently by sharing information along so called interface conditions between neighboring sub-systems. These interface conditions can be optimized for the performance of the algorithm [5]. Unfortunately, these methods might suffer from convergence problems, and suitable preconditioning techniques lead to an additional computational cost.

Multigrid methods (MGM) are known to be a viable alternative to the previous solution strategies especially for elliptic dominated problems [6]. They are the fastest numerical methods for solving boundary value problems [7]. Multigrid methods were the first to overcome the complexity barrier connected with that the amount of work does not remain proportional to the number of unknowns. The starting point of the multigrid and indeed also its ultimate upshot is the following "golden" rule: The amount of computational work should be proportional to the amount of real physical changes in the computed system.

The field of multigrid methods has become too large to review in a single article. Therefore, in this paper, we restrict our attention to the class of problems which is actual one for fluid dynamics: Navier-Stokes equations and convection- diffusion problems.

2. Multigrid Method: Main Development Stages

First working multigrid method was developed and analyzed by Fedorenko [8] for the Laplace equation on the unit square. Bachvalov [9] considered the theoretically much more complex case of variable coefficients.

The main observation of multigrid techniques is based on a Fourier analysis of the residual (or error) vector of a sequence of iterates that are generated by a scheme such as Jacobi or Gauss-Seidel (for instance). This means that these residual vectors are analyzed in the eigen-basis associated with the iteration matrix M - assuming that M has a complete set of eigenvectors. In the case of Jacobi, the observation is that the components associated with the largest eigenvalues (in the original matrix) will decrease rapidly. However, those associated with the smallest eigenvalues will converge much more slowly. As a result after a few steps, the

"high-frequency" components may have converged while the "low-frequency" components may have made very little progress in comparison. To correct this situation, researchers developed methods that used several grids. The simplest idea is to use two meshes one fine and one that is coarser, where the fine mesh can be viewed as the result of refining the coarse one. The iteration initially takes place on the fine mesh. After a few steps, the residual is projected onto the coarse mesh, by some form of restriction. Let A_{2h} be the matrix for the problem on the coarse mesh and r_{2h} this projected residual. The system $A_{2h}\delta = r_{2h}$ is then solved on the coarse mesh by means of a few steps of relaxation. This is called a correction step. The vector δ is then extrapolated into the finer mesh and the result is added as a correction to the iterate on the fine mesh.

Although the basic idea of combining discretization on different grids in an iterative scheme appears to be very natural, the potential of this idea was not recognized before the middle of the 1970s. At this time, the multigrid idea began to spread. The report of Hackbusch [10] and the paper of Brandt [11] were the historical breakthrough. The first big multigrid conference in 1981 in Koln was a culmination point of the development; the conference proceedings edited by Hackbusch and Trottenberg [12] are still a basic reference. With Hackbusch's 1985 monograph [13], the first stage in multigrid theory came to an end.

Russian scientists have made great contribution to the development of the MGM method. The Russian Federation National Award 2003 was handed over to G. Astahancev, N. Bahvalov, R. Fedorenko, V. Shaidurov for a cycle of fundamental works on creation and the subsequent heading highly effective multigrid method for the numerical solution of a wide class of mathematical physics problems.

Today, multigrid methods are used in nearly every field where partial differential equations are solved by numerical methods. The multigrid method [14] belongs to a group of iterative solvers, and it is one of the most efficient and widespread methods to solve large systems of linear equations [15]. Its efficiency is based on the fact that the multigrid method presents a potential to solve N×N linear systems with only O(N) computational effort.

Two different approaches can be accomplished employing the multigrid method according to the kind of data and information employed and also how the operators deal with them: the geometric multigrid (GMG) and the algebraic multigrid (AMG). The main difference between AMG and GMG is related to the manner of constructing the coarser grids [16]: the AMG method requires no knowledge of the geometry of the problem [17]. The GMG method [14] employ fixed grid hierarchies and, therefore, an efficient interplay between smoothing and coarse-grid correction has to be ensured by selecting appropriate smoothing processes [14]. On the other hand, the AMG method [16, 17] fixes the smoother to some simple relaxation scheme and enforces an efficient interplay with coarse-grid correction by choosing the suitable coarser levels and the interpolation [14]. The grid hierarchies in the AMG is generated in the setup phase (which is an initial or start-up phase), by considering the coefficient matrix, as well as the build interpolation, restriction and coarse-grid operators [17, 18]. The application of the AMG method includes problems in which the use of the GMG method is difficult or even impracticable, such as: unstructured grids, large matrix equations which are not at all derived from continuous problems, extreme anisotropic equations and so on. A remarkable use of the AMG method takes place when there is none information about the problem geometry [18]. Table 1, adapted from Chang et al. [19], provides a comparison between the AMG and the GMG methods.

Both strategies, GMG and AMG, present similar steps. The first one is the generation of the coarser grids (levels), which is followed by the transfer of information among different

Table 2.1: Comparison between the GMG and the AMG methods.

Features	GMG	AMG
Solved problem	Continuous problem	Linear systems of algebraic equations
Used information	Geometrical structure of the problem	Only entries of the matrix
Program	Necessity of composing a program for each problem	Only one program for different problems
Efficiency	Very good	Good

grids (restriction and prolongation operators). Then linear system in each grid is solved by an iterative smoother (solver), according to the choice of a multigrid cycle (F-cycle, V-cycle, W-cycle) [14]. The decision about choice of iterative method as smoother (solver), operators restriction and prolongation often involve considerable algorithmic research.

According to Trottenberg et al. [14], a single modification in the algorithm might result in a significant reduction of the CPU time requirements. The efficiency of the multigrid methods is also related to the adaptations of the multigrid components, which should be made properly according to the underlying physical problem and the variational formulation [20]. Unfortunately, the works available in the literature do not present deep studies about the components of the AMG algorithm and their optimization. More precisely, in such works, new coarsening algorithms and/or new interpolation operators are introduced, like the work of Xiao et al. [21], or to papers in which the AMG performance is compared to the GMG one [20,22]. In the latter case, such comparisons are limited to the CPU time, the number of cycles and the study of the multigrid efficiency provided by the speedup value.

The CPU time and its growth, according to the number of unknowns, were studied by Watanabe et al. [22], for both AMG and GMG. Both multigrid methods were also studied by Langer and Pusch [20], where comparisons for the number of cycles spent by the AMG and by the GMG, as well as the time requirements for the auxiliary grids generation were presented. The number of cycles was also revised by Wu and Elman [23], by using as stop criterion a given tolerance value; it was seen that the GMG convergence was slower than the AMG for convection-diffusion problems. Campos et al. [24] made a comparison between the performances of the AMG and the GMG, both with parallelized and preconditioned algorithms which are suitable for a non-linear system of differential equations. Additionally, simulations were performed varying the number of grids, for both the AMG and the GMG, and the grid reduction factor for the AMG, executed by 1, 2, 4 or 6 processors; the performance of the AMG algorithm was better than the one obtained by the GMG for both CPU time and memory.

Systematic studies about the multigrid parameters where found only for the GMG method. Gaspar et al. [25] presented theoretical and numerical results for the GMG with triangular grids, by applying distinct multigrid cycles, different numbers of inner iterations and proposing a new smoother (solver). On the other hand, Oliveira et al. [26] computed optimum values for the inner iterations and the levels used in heat diffusion problems with structured square grids. As it was mentioned above MGM is very actively used in computational fluid dynamics.

Computational fluid dynamics (CFD) gives rise to very large systems requiring efficient solution method. Not surprisingly, MGM found application in CFD at an early stage. The compressible potential equation was solved with MGM in 1976, the incompressible Navier-Stokes equations shortly after [27,28].

3. Basic Principles of the MGM

Multigrid method is not a fixed algorithm. There is rather a multigrid technique fixing only the framework of the algorithm. The efficiency of the multi-grid algorithm depends on the adjustment of its components to the problem under consideration [13]. Thanks to this fact the main contemporary study on multigrid methods focuses on "almost" self-adjoint positive-definite problems in linear case, and on "almost" linear problems in nonlinear case.

There are many variations of multigrid algorithms, but the common feature is that a hierarchy of discretizations (grids) is considered. The important steps are: smoothing - reducing high frequency errors, for example using a few iterations of the Gauss-Seidel method; restriction - downsampling the residual error to a coarser grid; interpolation or prolongation - interpolating a correction computed on a coarser grid into a finer grid.

The multigrid idea is based on two principles - error smoothing and coarse grid correction:

- the principle of smoothing - many classical iterative methods similar to the method of Gauss-Seidel or Jacobi has the effect of smoothing error when applied to discrete problems;
- the principle of coarse-grid correction - smooth components of the error can be well represented on a coarser grid where the decision is less expensive.

Consider the two-grid cycle for discrete linear boundary value problem

$$L_h u_h = f_h \qquad (\Omega_h). \tag{3.1}$$

- Smoothing procedure
  - Compute $\bar{u}_h^n$, applying ν_1 iterations of the smoothing method
- Coarse-grid correction
  - Compute residial $d_h = f_h - L_h \bar{u}_h^n$
  - Restrict residial $d_{2h} = R_h^{2h} d_h$
  - Solve coarse grid equation $L_{2h} v_{2h} = d_{2h}$
  - Prolongate coarse grid solution $v_h = P_{2h}^h v_{2h}$
  - Correct the required solution $u_h^n = \bar{u}_h^n + v_h$
- Post-smoothing
  - Compute u_h^{n+1}, using ν_2 iterations of the smoothing method

Using this notation, the operator of two-grid method can be written as following

$$M_h^{2h} = S_h^{\nu_2} K_h^{2h} S_h^{\nu_1}, \tag{3.2}$$

where $K_h^{2h} = I_h - P_{2h}^h L_{2h}^{-1} R_h^{2h} L_h$ - operator of the coarse-grid correction, S_h - smoothing operator.

Multigrid methods are based on the recursive use of a two-grid scheme. A basic two-grid method combines the action of a smoother, often a simple iterative method such as Gauss-Seidel, and a coarse grid correction, which involves solving a smaller problem on a coarser grid.

A V-cycle multigrid method is obtained when this coarse problem is solved approximately with 1 iteration of the two-grid scheme on that level, and so on, until the coarsest level on which an exact solve is performed. Other cycles may be defined, for instance the W-cycle based on two stationary iterations at each level, see, e.g., [14].

4. MGM for Navier-Stokes Equations

Over the last two decades, many methods have been developed to solve the incompressible unsteady Navier-Stokes equations. Among those, the pressure based method, penalty method, and pseudo-compressibility method are the most successful approaches [28]. The pressure-based methods represented by the SIMPLE-family codes developed by Patankar et al. were the dominant approaches in simulation of incompressible flow and compressible subsonic flow during the 1970s and 1980s.There are many industrial codes which use this kind of approach. Some of them have developed to handle transonic and supersonic flows.

The multigrid is one of the most powerful numerical method for improving the efficiency of computational fluid dynamics solvers. It is broadly used for solving incompressible unsteady Navier-Stokes equations [29].Classical multigrid method have been proved to be extremely efficient on solving pressure Poisson equation, enabling solution to the level of discretization errors in just a few minimal work units, so that the total work invested in the solution grows linearly with the number of variable flow, such as pre-optimization techniques which accelerate the multigrid process before the coarse grid procedure.

As Reynolds number is high, the numerical calculation of Navier-Stokes equations has the convergence problem. Michael [30] has solved the Navier-Stokes equation by using pressure-correction multigrid Newton precondition method. He's got numerical results that Reynolds number is just 1.0. Ping [31] has showed that the sequential regularization method can solve Navier-Stokes equation with high Reynolds number. But, like penalty method, the convergence is slow when we use sequential regularization method to solve Navier-Stokes equation [31].

A hybrid multigrid method was suggested for the unsteady incompressible Navier-Stokes equations [32]. This approach is presented for the high Reynolds incompressible flow, based on multigrid method and sequential regular method. In [32] the velocity-pressure increment and sequential regular equations are derived from the Navier-Stokes equation. The pressure increment optimum sweep algorithm is discussed. The numerical results of the close square flow show that this algorithm improves the convergence rate for high Reynolds unsteady flow.

We propose once more approach for solving the Navier-Stokes equations in hydrodynamics problems.

Consider classical formulation of the Navier-Stokes equation in domain $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f_1,\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = f_2,\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = f_3,
\end{aligned}$$

$$u(x,y,t) = g_1(x,y,t), \quad v(x,y,t) = g_2(x,y,t) \quad \text{on} \quad \partial\Omega,$$
$$u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y).$$

where Re is Reynolds's number, and $V = (u(x,y,t), v(x,y,t))$ is the velocity, P is the pressure.

The system of Navier-Stokes two-dimensional equations for two components of velocity and pressure for an incompressible viscous fluid is considered. To approximate the time derivative and inertial first space derivatives the method of characteristics is used [33]. Space discretization is carried out by finite element method. This mixed method of approximation was suggested by O. Pironneau in the 1980-th year and it's more often used in numerical researches now [34]. It's used a mixed formulation in the finite element method, when a combination of simple finite elements - bilinear for velocities and constant elements for pressure is applied. This combination provides stability of pressure calculation with additional application of a numerical filtration.

After discretization we obtain a linear algebraic equation system with a symmetric matrix which has a spectrum with alternating signs. We use multigrid method for solving this system. For the Navier-Stokes equations it has been shown that by mixing the method of characteristics and the finite element method we are able to derive first and second order accurate conservative schemes of the upwinding type.

Application of the method of characteristics and the finite element method combination allows building the effective numerical algorithm. These schemes are numerically better than the usual upwinding schemes because they require numerical solution of symmetric systems only. After discretization we obtain a linear algebraic equation system with a symmetric matrix which has a spectrum with alternating signs [35]. We use multigrid method with simple iteration method as the smoother for solving this system.

The results of some numerical experiments allow to conclude the efficiency of the suggested approach for solving the Navier-Stokes equations.

5. MGM for Convection-Diffusion Problems

As mentioned above the other most common equation in the computational fluid dynamics field is the convection- diffusion equation. Mathematical models that involve a combination of convective and diffusive processes are among the most widespread in all the sciences. Research of these processes is especially important and difficult when convection is dominant. At the same time, convection-diffusion equations are used as tests in researching iterative methods for solving systems of strongly nonsymmetric linear equations.

Analysis of algebraic multigrid parameters for two-dimensional steady-state heat diffusion equations was presented in [36]. In this work it is provided a comparison for the algebraic multigrid (AMG) and the geometric multigrid (GMG) parameters, for Laplace and Poisson two-dimensional equations in square and triangular grids. The analyzed parameters are the number of: inner iterations in the solver, grids and unknowns. For the AMG, the effects of the grid reduction factor and the strong dependence factor in the coarse grid on the necessary CPU time are studied. For square grids the finite difference method is used, and for the triangular grids, the finite volume one. The results are obtained with the use of an adapted AMG1R6 code of Ruge and Stuben. For the AMG the following components are used: standard coarsening, standard interpolation, correction scheme (CS), lexicographic Gauss-Seidel and V-cycle. Comparative studies among the CPU time of the GMG, AMG and singlegrid are made. It was verified that: (1) the optimum inner iterations is independent of the multigrid, however

it is dependent on the grid; (2) the optimum number of grids is the maximum number; (3) AMG was shown to be sensitive to both the variation of the grid reduction factor and the strong dependence factor in the coarse grid; (4) in square grids, the GMG CPU time is 20% of the AMG one.

Modification of algebraic multigrid for effective GPGPU-based solution of nonstationary hydrodynamics problems was presented [37]. The modification is easy to implement and allows us to reduce number of times when the multigrid setup is performed, thus saving up to 50% of computation time with respect to unmodified algorithm.

As it's mentioned above algebraic multigrid (AMG) [14] is one the most effective methods for solution of large sparse unstructured systems of equations, arising, for example, from discretizations of elliptic differential equations. AMG applies ideas of geometric multigrid (smoothing and correction on coarse grid) to solution of certain classes of algebraic systems of equations. The main advantage of AMG (besides robustness and efficiency) is its ability to solve elliptic partial differential equations discretized on unstructured grids [38]. AMG can be used as a black-box solver for various computational problems, since it does not require any information about underlying geometry. This fact makes GPGPU-based implementation of AMG extremely attractive [39].

The algorithm of AMG has two major stages: setup phase and solution phase. The setup phase in classic formulation of AMG is very hard to parallelize because of its intrinsic serial nature.To the contrary, the solution phase allows straightforward parallelization.

When GPGPU technique is applied to the solution phase, tenfold acceleration rate is easily achieved [40]. However, according to Amdahl's law, we cannot exceed acceleration rate of $3\times$ for solution of single system of equations, since about 30% of computational work belongs to the setup phase. In [37] it was shown that the restriction may be loosened for solution of non steady problems with constant or slowly changing coefficients.

The improved algorithm was successfully applied to the solution of several hydromechanics problems. In particular, it was presented substantial acceleration rate for several oil reservoir simulation problems.

A hybrid multigrid method for convection-diffusion problems is considered in [6]. Presented approach is an advanced performance study of a multigrid method designed for convection-diffusion problems developed in Khelifi et al. [41]. The proposed scheme with the separation of the operators enables an individual treatment for each operator: while the piecewise constant operator is used for the convective part, each off-diagonal entry of the coarse diffusion operator is scaled by a geometric factor. Numerical examples illustrate the fast convergence and the outstanding robustness of the proposed method, compared to other known methods.

The increasing complexity in the geometry and the physical modeling of today's simulation problems makes the use of geometric multigrid more difficult. Algebraic multigrid (AMG) applies multigrid principles without requiring any information about the geometry [38]. The construction of the hierarchy of grids is fully automatic. The levels are created using purely algebraic entities such as the matrix entries. Algebraic multigrid is well developed especially for symmetric positive definite problems and diagonally dominant M-matrices arising from the discretization of second-order scalar elliptic partial differential equations. Nevertheless, they were successfully used in non symmetric and more general cases without any modification [18], [38].

Hence, the solution of a large class of linear systems arising from the discretization on unstructured meshes of a scalar PDE became possible thanks to such a procedure. Among the different AMG approaches we can cite the classical AMG [38] (often referred to as the

Ruge-Stuben approach), the AMGe (algebraic multigrid for finite elements), the element free AMGe [42], adaptive methods and aggregation [43]. The latter can be regarded as a limiting case of the Ruge-Stuben multigrid procedure where every fine point is interpolated from a unique coarse point [38]. The prolongation operator associated to this approach is simply the piecewise constant interpolation (P0). This approach is well adapted to the finite volume discretization widely used in CFD applications as it consists of creating coarse mesh cells by agglomeration of some fine mesh cells. The finite volume approach copes naturally with the complex polyhedral cell geometries produced by the aggregation procedure, in contrast to the finite element methods, which are difficult to apply on arbitrary polyhedral cells, as they require the definition of shape functions for each element geometry. Aggregation is also consistent with the finite volume philosophy as the integral of a function on an aggregate is equal to the sum of the integrals on the cells composing this aggregate. For these reasons, aggregation seems to be a judicious choice for CFD problems.

However, it is well known that the standard aggregation based algebraic multigrid method is not suitable for elliptic second order problems. It fails to ensure the condition for the optimal multigrid efficiency [44]. Many techniques exist to overcome the slow convergence rate of the straightforward aggregation scheme and recover the theoretical convergence.

The rescaling of the operator can be an effective remedy. The simplest method consists of the rescaling by a global number. The number is usually simply computed and the method is quite easy to implement but it is limited to a geometric multigrid procedure. In [45], another approach called "smoothed aggregation" is developed. It improves the convergence of the multigrid procedure but the inconvenience is that it increases the number of the off-diagonal entries of the coarse level matrix.

Suggested method is based on a third approach, studied in the finite volume context, with a discretization on arbitrary polyhedra.It maintains the simplicity of the P0 interpolation with a little extra-work consisting of a face based rescaling [41]. This approach produces a coarse diffusion problem that mimics the direct discretization of the diffusion operator on the coarse mesh. However, in the presence of the convection operator in addition to the elliptic one, the optimal use of multilevel techniques is more difficult to establish. Yet, several different strategies for overcoming these difficulties exist [41, 43, 50]. In [6] the performances of the multigrid procedure developed in [41] is studying, with a comparison to more well known Krylov methods in terms of number of iterations and wall clock time (WCT) time.

6. MGM for Convection-Diffusion Problems with Dominant Convection

We suggest modification of MGM for convection-diffusion problems with dominant convection.

Consider the model problem in the domain $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned} &\frac{1}{2}\sum_{k=1}^{2}\left(v_k(x)\frac{\partial u(x)}{\partial x_k} + \frac{\partial\left(v_k(x)u(x)\right)}{\partial x_k}\right) - \frac{1}{Pe}\sum_{k=1}^{2}\frac{\partial}{\partial x_k}\left(\frac{\partial u(x)}{\partial x_k}\right) = f(x), \quad x \in \Omega \\ &u(x) = 0, \qquad x \in \partial\Omega, \end{aligned} \tag{6.1}$$

where $x = (x_1, x_2)$. Eq. (6.1) has a small parameter at the highest derivative. We consider incompressible environments, so $divV = \sum_{k=1}^{2} \frac{\partial v_k}{\partial x_k} = 0$.

The finite difference method with central differences is used to discretize (6.1). We obtain an algebraic system of linear equations with a strongly nonsymmetric matrix:

$$L_h u_h = f_h, \tag{6.2}$$

which is considered on a grid

$$G_h = \left\{ (x_1, x_2) : x_k = ih, h = \frac{1}{n}, n \in \mathbb{N}, k = 1, 2, i \in \mathbb{Z} \right\},$$

where u_h and f_h are the grid functions on G_h, and L_h is the linear operator

$$L_h : E(G_h) \to E(G_h),$$

with $E(G_h)$ being the linear space of grid functions defined on G_h.

Corresponding to the operator L_h, the matrix L is a strongly sparse nonsymmetric one without diagonal dominance. Split the matrix L as a sum of symmetric and non-symmetric parts:

$$L = L_0 + L_1,$$

where

$$L_0 = \frac{1}{2}(L + L^*) = L_0^*, \quad L_1 = \frac{1}{2}(L - L^*) = -L_1^*$$

being the symmetric part and the skew-symmetric part of the matrix L. In some matrix norm it holds that

$$\|L_0\|_* \ll \|L_1\|_*.$$

The matrix L is a real positive one. It means that its symmetric part is positive definite:

$$L_0 = \frac{1}{2}(L + L^*) = L_0^* > 0.$$

Split the matrix L_1 as

$$L_1 = K_l + K_u \quad \text{and} \quad K_u = -K_l^*,$$

where K_l and K_u are respectively, the lower and the upper triangular parts of the skew-symmetric matrix L_1.

To solve the linear system (6.2), we suggest to use the MGM, where the triangular iterative method (TIM) [46–49] will be used as the smoother of the MGM [50].

Consider the structure of the TIM. Any iterative method can be written in the standard form

$$B\frac{u_{n+1} - u_n}{\tau} + Lu_n = f, \quad n = 0, 1, 2, \ldots. \tag{6.3}$$

The choice of the operator B defines the triangular skew-symmetric iterative method. For a standard TIM, the operator B is constructed as follows:

$$B = E + 2\tau K_l \quad \text{or} \quad B = E + 2\tau K_u. \tag{6.4}$$

For TIM1,

$$B = \alpha E + 2K_l \quad \text{or} \quad B = \alpha E + 2K_u, \tag{6.5}$$

and for TIM2,

$$B = \alpha_i E + 2K_\ell \quad \text{or} \quad B = \alpha_i E + 2K_u, \tag{6.6}$$

where τ is a scalar parameter.

Parameters of the offered methods $\alpha_i, \alpha > 0$ get out under formulas:

$$\alpha = \|M\|,$$
$$\alpha_i = \sum_{j=0}^{n} |m_{ij}|, \quad i = 0, 1, \ldots, n,$$

where $M = \{m_{ij}\}_0^n$ is a symmetric matrix constructed by the following way:

$$M = A_0 + K_u - K_l,$$

and n is the dimension of the matrix L.

Any method in this class has the same behavior as the Gauss-Seidel iteration, i.e., it quickly reduces the high-frequency, but not low-frequency components of error frequencies. This is the necessary property of the smoother of MGM, that is why we have used these methods as the smoothers.

The convergence of proposed MGM modifications with triangular skew-symmetric smoothers is discussed in [50].

7. Model Problem

We consider the problem (6.1) to research properties of MGM modifications with suggested smoothers. We research four model problems with different velocity fields, presented in table 7.1.

Table 7.1: Velocity coefficients for test problems.

Problem N	v_1	v_2
1	1	-1
2	$1 - 2x$	$2y - 1$
3	$x + y$	$x - y$
4	$\sin 2\pi x$	$-2\pi y \cos 2\pi x$

The different Peclet numbers were considered: Pe=10, 100, 1000, 10000.

The problem (6.1) was solved by multigrid method modifications with three kinds of smoothes: TIM, TIM1 and TIM2. The number of smoothing iterations in MGM is 15. In table 7.2 the results of comparison of the suggested MGM modifications with triangular skew-symmetric methods and Gauss-Seidel method as the smoothers on a grid 33×33 are presented. The symbol D means, that on the given problem a method doesn't converge.

The numerical results show that: a) the suggested multigrid method modification with triangular iterative smoothers proved to be effective for solving convection-diffusion problems with dominant convection; b) the multigrid method with the smoothers TIM1 and TIM2 is more effective for these problems, than MGM with TIM. Under consideration the most effective method for convection - diffusion problems with dominant convection is MGM with smoother TIM2; c) the coefficient of skew-symmetry κ= Pe*h*$|v|/2$ has the greatest influence on the behaviour of the method (neither the size of grid nor the coefficients of equation separately).

8. Conclusion

Originally introduced as a way to numerically solve elliptic boundary-value problems, multigrid methods, and their various multiscale descendants, have since been developed and applied

Table 7.2: MGM iteration number and CPU-time on the grid 32×32.

Pe	MGM (Seidel)	MGM (TIM)	MGM (TIM1)	MGM (TIM2)	κ= Pe*h*$\lvert v\rvert/2$
Problem 1: $v_1(\mathbf{x}) = 1$ $v_2(\mathbf{x}) = -1$					
10	13	35	30	30	0,009765
	0:00:31	0:00:94	0:00:93	0:00:109	
100	63	7	5	5	0,097656
	0:00:188	0:00:16	0:00:15	0:00:15	
1000	D	13	9	9	0,976562
		0:00:31	0:00:47	0:00:31	
10000	D	78	58	58	9,765625
		0:00:250	0:00:203	0:00:188	
Problem 2: $v_1(\mathbf{x}) = 1 - 2x_1$ $v_2(\mathbf{x}) = 2x_2 - 1$					
10	22	72	53	50	0,009765
	0:00:62	0:00:188	0:00:172	0:00:171	
100	18	24	19	14	0,097656
	0:00:47	0:00:63	0:00:63	0:00:47	
1000	D	16	12	6	0,976562
		0:00:47	0:00:31	0:00:15	
10000	D	59	51	32	9,765625
		0:00:187	0:00:171	0:00:109	
Problem 3: $v_1(\mathbf{x}) = x_1 + x_2$ $v_2(\mathbf{x}) = x_1 - x_2$					
10	16	43	35	34	0,019531
	0:00:47	0:00:125	0:00:110	0:00:110	
100	23	9	7	5	0,195312
	0:00:62	0:00:31	0:00:15	0:00:15	
1000	D	17	12	8	1,953125
		0:00:47	0:00:31	0:00:31	
10000	D	74	55	36	19,53125
		0:00:219	0:00:187	0:00:125	
Problem 4: $v_1(\mathbf{x}) = \sin 2\pi x_1$ $v_2(\mathbf{x}) = -2\pi x_2 \cos 2\pi x_1$					
10	17	39	32	27	0,061359
	0:00:47	0:00:109	0:00:110	0:00:94	
100	D	16	12	7	0,613592
		0:00:47	0:00:47	0:00:31	
1000	D	29	22	10	6,135923
		0:00:94	0:00:78	0:00:31	
10000	D	193	159	57	61,35923
		0:00:625	0:00:562	0:00:187	

to various problems in many disciplines. Two different approaches can be accomplished employing the multigrid method according to the kind of data and information employed and also how the operators deal with them: the geometric multigrid (GMG) and the algebraic multigrid (AMG).

MGM is very actively used in computational fluid dynamics.For the Navier-Stokes equations it has been shown that by mixing the method of characteristics and the finite element method we are able to obtain first and second order accurate conservative schemes of the upwinding type.

The multigrid algorithms are immediately useful for simple linear elliptic problems on simple

domains. With relatively minor modifications, they can be generalized to handle nonlinear, anisotropic, or moderately inhomogeneous problems as well as elliptic systems. The perspective matrix splitting iteration methods, e.g., the Hermitian and skew-Hermitian splitting (HSS) iteration method [51] and its variuos generalizations and variants [52], [53] can be used as effective smoothers of the multigrid methods for solving nonself-adjoint positive-definite linear problems. Still more elaborate modifications are required to obtain efficient solvers for singular perturbation problems and for problems with discontinuous coefficients, complex domains, or unstructured grids. These topics have been researched widely over the last 30 years. Researchers have resolved many of the difficulties, while others remain open and subject to research.

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