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USING THE SKEW-SYMMETRIC ITERATIVE METHODS FOR SOLUTION OF AN INDEFINITE NONSYMMETRIC LINEAR SYSTEMS*

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Abstract

The concept of the field of value to localize the spectrum of the iteration matrices of the skew-symmetric iterative methods is further exploited. Obtained formulas are derived to relate the fields of values of the original matrix and the iteration matrix. This allows us to determine theoretically that indefinite nonsymmetric linear systems can be solved by this class of iterative methods.

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Key words: Skew-symmetric iterative methods, Indefinite nonsymmetric linear systems.

1. Introduction

Consider a system of linear equations

$$Au = f, \tag{1.1}$$

where A is an $N \times N$ matrix, $u = (u_1, ..., u_N)^T$ is the vector of solution, $f = (f_1, ..., f_N)^T$ is the vector of the right-hand side. The system of linear equations (1.1) can be solved by the following iterative method [9]:

$$B(\omega)\frac{y_{k+1} - y_k}{\tau} + Ay_k = f, \tag{1.2}$$

where $A, B(\omega)$ are nonsingular matrices of size $N \times N$ in a finite-dimensional Euclidean space. In the following $B(\omega)$ will be shortly denoted by B, whenever there is no possible confusion. We rewrite (1.2) as

$$y_{k+1} = Gy_k + \tilde{f}, \quad \tilde{f} = \tau B^{-1} f,$$

where

$$G = I - \tau B^{-1} A \tag{1.3}$$

is the corresponding iteration matrix.

Let λ_j be an eigenvalue of the matrix $F = B^{-1}A$, μ_j be an eigenvalue of G, and $Sp(G)$ be the set of all eigenvalues (the spectrum) of G in (1.3). Investigation of convergence of iterative methods can be used in two ways:

(i) Spectral estimate, where we research spectrum of G, [12] and require that

$$|\mu_j| < 1, \quad j = 1, ..., n; \tag{1.4}$$

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(ii) Energy-norm estimate, where we check norm of G, [9] and demand that

$$\|G\|_D < 1, \tag{1.5}$$

with D being Hermitian positive definite matrix. The first one provides necessary and sufficient conditions and the second one only sufficient conditions.

Using (1.4) we have

$$\mu_j = 1 - \tau\lambda_j, \quad \tau > 0,$$

and, according to [12], a necessary and sufficient condition for the convergence of the iterative method is

$$|1 - \tau\lambda_j| < 1. \tag{1.6}$$

Let

$$\Re(\lambda) = \min_j Re(\lambda_j), \quad \rho^2 = \max_j |\lambda_j|^2.$$

Then by taking into account the introduced notation, from (1.6) we have

$$\tau < \frac{2\Re(\lambda)}{\rho^2}. \tag{1.7}$$

See [7] for details.

Definition 1.1. *Matrix F is named positive stable if $Re(\lambda_j(F)) > 0, \forall j = 1, \ldots, n$.*

Theorem 1.1. *If the matrix $F = B^{-1}A$ is positive stable, then the iterative method* (1.2) *converges if the condition* (1.7) *holds.*

To investigate convergence of iterative methods we have to know eigen-property about matrix F, a product of the matrices A and B, which determinates the iterative method. It was shown above that if F is positive stable, then the method converges. For example, if the matrix A is positive real and the matrix B is symmetric positive definite, then from Lyapunov theorem [3] we know that F is a positive stable matrix. So, we obtain convergence for a wide class of iterative methods (Jacobi, SSOR, CG and so on).

Energy-norm [9] and spectral [12] sufficient conditions were previously obtained for convergence of skew-symmetric iterative method in [6] and [2], when the initial matrix A is positive real. If the matrix A losses this property, the question about sufficient conditions of convergence remains open. Unfortunately, there is no discussion about the convergence in the sense of spectral or energy-norm. We propose to use field-of-values to solve this problem.

The field-of-values [3] of the matrix A is defined as

$$H(A) = \{x^*Ax : x \in \mathcal{C}^n, x^*x = 1\}.$$

It is known [4] that

$$Sp(A) \subseteq H(A).$$

We will begin with short description of skew-symmetric iterative methods (SSM), then we will show that the field-of-values works for investigating the convergence of SSM, using formulas from [7], which connect fields-of-values of the matrices A, B, their symmetric part A_0 and skew-symmetric part A_1, and eigenvalues of the iteration matrix G. These formulas will be used to derive sufficient conditions for convergence about matrix A when it is not positive real.

2. Skew-Symmetric Iterative Methods

The matrix A can be naturally expressed as a sum of a symmetric matrix A_0 and a skew-symmetric matrix A_1. If A_0 is positive definite, then the matrix A is named positive real. We split matrix A_1 into a sum of triangular matrices K_U and K_L and establish [5] the classes of triangular and product triangular skew-symmetric iterative methods.

The triangular skew-symmetric iterative methods [7] are defined in (1.2) with the matrix being chosen as

$$B = B_c + \omega((1+j)K_L + (1-j)K_U), \quad j = \pm 1, \; B_c = B_c^*. \tag{2.1}$$

The product-triangular skew-symmetric iterative methods [2] are defined in (1.2) with the matrix B being chosen as:

$$B = (B_c + \omega K_L)\, B_c^{-1}\, (B_c + \omega K_U)\,. \tag{2.2}$$

In (2.1) and (2.2) B_c is a symmetric matrix and ω is a real scalar parameter.

More general idea of the symmetric and skew-symmetric splitting can be used if the matrix A in (1.1) is complex [10]:

$$A_1 = K_L + H_0 + K_U - H_0 = \widetilde{K}_L + \widetilde{K}_U, \tag{2.3a}$$

$$\widetilde{K}_L = K_L + H_0,\, \widetilde{K}_U = K_U - H_0, \tag{2.3b}$$

where H_0 is a Hermitian matrix. An iterative method based on this splitting has been proposed in [1], [11]. Unfortunately, there is a problem with the inverse of the matrix B.

We write the matrix in (2.1) and (2.2) as a sum of symmetric and skew-symmetric components:

$$B = B_0 + B_1, \quad B_0 = \frac{1}{2}(B + B^*), \quad B_1 = \frac{1}{2}(B - B^*).$$

Main property of skew-symmetric iterative method is

$$B_1 = \omega A_1. \tag{2.4}$$

Thus, for both triangular and product-triangular skew-symmetric methods the skew-symmetric parts of A and B coincide up to a parameter. This property is crucial and will be extensively used.

3. Convergence of Skew-Symmetric Iteration Methods

The problem of the convergence of the iterations can be reduced to determine the eigenvalues of the matrix F, i.e.,

$$Fx = \lambda x \;\Leftrightarrow\; B^{-1}Ax = \lambda x \;\Leftrightarrow\; Ax = \lambda Bx. \tag{3.1}$$

Moreover, in order to check whether the method converges we do not have to know the whole spectrum of the matrix F. It is sufficient to check only whether $Re(\lambda) > 0$.

Notice that the matrices A and B in (3.1) are real and their eigenvectors and eigenvalues are, in general, complex. Denote the numerical range of a matrix Ψ by $R(\Psi)$. Let

$$\alpha = \alpha(A_0) = \frac{(A_0x, x)}{(x, x)} \in R(A_0), \quad \forall x \neq 0, \tag{3.2a}$$

$$\beta = \beta(B_0) = \frac{(B_0x, x)}{(x, x)} \in R(B_0), \quad \forall x \neq 0, \tag{3.2b}$$

$$i\gamma = i\gamma(A_1) = \frac{(A_1x, x)}{(x, x)} \in R(A_1), \quad \forall x \neq 0. \tag{3.2c}$$

The sets $R(A_0)$ and $R(B_0)$ are real, but the set $R(A_1)$ is imaginary, [3]. Since A_0, B_0 and A_1 are normal [3], their numerical ranges are bounded by their extremal eigenvalues, i.e.,

$$R(A_0) \in [\alpha_0, \alpha_1], \qquad \alpha_0 = \lambda_{min}(A_0), \quad \alpha_1 = \lambda_{max}(A_0), \tag{3.3a}$$

$$R(B_0) \in [\beta_0, \beta_1], \qquad \beta_0 = \lambda_{min}(B_0), \quad \beta_1 = \lambda_{max}(B_0), \tag{3.3b}$$

$$R(A_1) \in [-i\gamma_1, i\gamma_1], \ \gamma_1 = |\lambda_{max}(A_1)|. \tag{3.3c}$$

Using simple calculation, we have

$$\lambda_0 = \frac{\omega\gamma^2 + \alpha\beta}{\omega^2\gamma^2 + \beta^2}, \quad \lambda_1 = \frac{\beta - \omega\alpha}{\omega^2\gamma^2 + \beta^2}\gamma, \tag{3.4}$$

where $\alpha \in R(A_0), \beta \in R(B_0), i\gamma \in R(A_1)$, and ω is the real parameter. See [7].

Parameters α, β, γ are some values of the numerical ranges of the corresponding matrices given in (3.2). It is a difficult task to determine these values. However, if their bounds are known we can obtain bounds for the values λ_0 and λ_1, i.e., the spectrum of F.

Does skew-symmetric methods converge if the matrix A is an indefinite matrix? Answer of this question can push us to investigating the field-of-value of the matrix F, which we have made above. We are interested in the following values

$$Ch1 = \omega\gamma^2 + \alpha\beta, \tag{3.5}$$

which can be taken under conditions (3.3). Clearly,

$$\begin{cases} \alpha \in [\alpha_0, \alpha_1], \\ \beta \in [\beta_0, \beta_1], \quad \beta_0 > 0, \\ \gamma^2 \in [\gamma_0^2, \gamma_1^2], \quad \gamma_0^2 \geq 0. \end{cases} \tag{3.6}$$

Note that the minimum eigenvalue of the matrix A_1 appears here as a bound for γ. The value of γ_0 depends on the dimension of A_1. If it is odd, then $\gamma_0 = 0$. In general, we have

$$\gamma_0 \geq 0.$$

If the signs of α_1 and α_0 coincide, then without loss of generality we can assume that $\alpha_1 > \alpha_0 > 0$. Otherwise we multiply the original linear system (1.1) by -1.

Expression (3.5) is a nominator in the relations for the real part of eigenvalues of $F = B^{-1}A$. Recall that B_0 is symmetric and B is chosen such that B_0 is positive definite, i.e., $\beta_0 > 0$. Clearly, if $\alpha_0 > 0$, then

$$Ch1 \in [\omega\gamma_0^2 + \alpha_0\beta_0, \omega\gamma_1^2 + \alpha_1\beta_1]. \tag{3.7}$$

Hence we obtain $\lambda_0 > 0$ and that the method converges.

Theorem 3.1. *Let A be positive real, the conditions* (2.4)*,* (3.2) *and* (3.3) *hold, and* $\beta_0 > 0$*. Then the iterative method* (1.2) *converges.*

So, in this case a positive real matrix A is multiplied by a positive real matrix B^{-1}, which yields the positive stable matrix F.

Now we consider the case

$$\alpha_0 < 0,$$

when the coefficient matrix A of the linear system is not positive real. (This property is very important for the convergence for a lot of iterative methods). See [8]).

Let

$$Ch1 \in [\omega\gamma_0^2 - |\alpha_0|\beta_1, \omega\gamma_1^2 + \alpha_1\beta_1]. \tag{3.8}$$

We are here interested in the lower bound $\omega\gamma_0^2 - |\alpha_0|\beta_1$. Let us assume that

$$\omega\gamma_0^2 - |\alpha_0|\beta_1 > 0,$$

and A_1 is nonsingular, i.e., $\gamma_0 > 0$. Then, if

$$\omega > \frac{|\alpha_0|\beta_1}{\gamma_0^2}, \tag{3.9}$$

we obtain the convergence of the iterative method as long as $\lambda_0 > 0$. As we see, the convergence can be achieved by choosing the parameter ω and requiring nonsingularity of the matrix A_1.

Theorem 3.2. *Let A and A_1 be nonsingular, the conditions* (2.4)*,* (3.2) *and* (3.3) *hold, and* $\beta_0 > 0$*. Then, if the matrix A is not positive real ($\alpha_0 < 0$ in* (3.6)*), the iterative method* (1.2) *converges as long as the condition* (3.9) *holds for the parameter ω.*

Thus, we have shown that the class of skew-symmetric iterative methods converge even under quite strict conditions on the coefficient matrix A of the linear system, i.e., even if A is not positive real.

4. Conclusion

The field-of-values is offered to be used for convergence investigation of skew-symmetric iterative methods for an indefinite system of linear equations.

It has been shown that skew-symmetric methods can be used to compute the solutions of a wide class of systems of linear equations with the matrices whose spectrum belongs to both left and right half planes.

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