ALTERNATELY LINEARIZED IMPLICIT ITERATION METHODS FOR SOLVING QUADRATIC MATRIX EQUATIONS[*]

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Abstract

A numerical solution of the quadratic matrix equations associated with a nonsingular M-matrix by using the alternately linearized implicit iteration method is considered. An iteration method for computing a nonsingular M-matrix solution of the quadratic matrix equations is developed, and its corresponding theory is given. Some numerical examples are provided to show the efficiency of the new method.

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1. Introduction

We consider the numerical solution of the following matrix equation

$$X^2 - BX - C = 0, \tag{1.1}$$

where $B, C, X \in R^{n \times n}$, all off-diagonal elements of B are nonnegative and C is a nonsingular M-matrix. The nonlinear matrix equation has numerous applications in control theory, signal processing and so on [1,2]. Some methods [3-6] have been developed extensively for solving the matrix equation. By simply transforming the quadratic matrix equation into an equivalent fixed-point equation, Bai et al. [3] constructed a successive approximation method and a Newton method based on the fixed-point equation. Higham and Kim [4] incorporated exact line searches into Newton method to solve the quadratic matrix equation.

Recently, Bai, Guo and Xu [7] proposed an alternately linearized implicit (ALI) iteration method for computing the minimal nonnegative solution of the algebraic Riccati equations (AREs). This method is more feasible and effective than the other methods. Applying this method, in this paper we propose a new numerical method for solving the quadratic matrix equation (1.1). The method for computing a nonsingular M-matrix solution of the quadratic matrix equation (1.1) is developed and its corresponding theory is given. Some numerical examples are provided to show the efficiency of the new method.

We first define some notations and introduce several basic results. For any matrices $A, B \in R^{n \times n}$, we write $A \geq B(A > B)$ if $a_{ij} \geq b_{ij}(a_{ij} > b_{ij})$ for all i, j. A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. It follows that any Z-matrix A can be written as the form $A = sI - B$, with s a positive real and B a nonnegative matrix. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B. It is called a singular M-matrix if $s = \rho(B)$; It is called a nonsingular M-matrix if $s > \rho(B)$. $\|A\|$ denotes the Frobenius norm of a matrix A.

Lemma 1.1. ([8]) *For a Z-matrix A, the following statements are equivalent:*

    (1) *A is a nonsingular M-matrix;*

    (2) *$A^{-1} \geq 0$;*

    (3) *$Av > 0$ for some vector $v > 0$.*

Lemma 1.2. ([7]) *Let*

$$K = \left[\begin{array}{cc} D & -C \\ -B & A \end{array} \right].$$

If K is a nonsingular M-matrix, then the algebraic Riccati equation (ARE)

$$\Re(X) = XCX - XD - AX + B = 0 \tag{1.2}$$

has a minimal nonnegative solution S, where A, B, C and D are real matrices of sizes $m \times m, m \times n, n \times m$ and $n \times n$, respectively.

Bai et al. [7] established a class of alternately linearized implicit (ALI) iteration methods for computing the minimal nonnegative solutions of the ARE (1.2) by the following algorithm.

Algorithm 1.1. ([7]) (The ALI iteration method)

1) Set $X_0 = 0 \in R^{m \times n}$.

2) For $k = 0, 1, \cdots$, until $\{X_k\}$ convergence, compute $\{X_{k+1}\}$ from $\{X_k\}$ by solving the following two systems of linear matrix equations:

$$\begin{cases} X_{k+\frac{1}{2}} (\alpha I + (D - CX_k)) = (\alpha I - A) X_k + B, \\ \left(\alpha I + \left(A - X_{k+\frac{1}{2}}C\right)\right) X_{k+1} = X_{k+\frac{1}{2}} (\alpha I - D) + B, \end{cases}$$

where $\alpha > 0$ is a given iteration parameter.

The ALI iteration method is better than both the Newton iteration method and the fixed-point iteration method.

We consider the numerical solution of the quadratic matrix equation (1.1) associated with a nonsingular M-matrix by using the ALI iteration method. The paper is organized as follows. First, the quadratic matrix equation (1.1) is transformed into the algebraic Riccati equations. Second, a numerical method for computing an M-matrix solution of the quadratic matrix equation is proposed in Section 2. Then, some numerical examples are provided in Section 3. Finally, conclusions are given in Section 4.

2. The Alternately Linearized Implicit Iteration Method

In order to solve the quadratic matrix equation (1.1), we will convert the quadratic matrix equation (1.1) into an algebraic Riccati equation.

By letting $X = \alpha I - Y$ as in [6], we can rewrite the equation (1.1) as

$$(\alpha I - Y)^2 - B(\alpha I - Y) - C = 0,$$

which gives

$$Y^2 - Y(\alpha I) - (\alpha I - B)Y + (\alpha^2 I - \alpha B - C) = 0, \tag{2.1}$$

where α is a real constant. Let

$$R = \begin{bmatrix} \alpha I & -I \\ -\alpha^2 I + \alpha B + C & \alpha I - B \end{bmatrix} \tag{2.2}$$

and $\alpha > 0$ be a parameter such that $\alpha^2 I - \alpha B - C \geq 0$, that is

$$\begin{cases} \alpha^2 - \alpha b_{ii} - c_{ii} \geq 0, \\ -\alpha b_{ij} - c_{ij} \geq 0, \end{cases} \tag{2.3}$$

where b_{ii} and c_{ii} are the i-th diagonal elements of B and C, and b_{ij} and $c_{ij}(i \neq j)$ are the off-diagonal elements of B and C, respectively. By means of (2.3), it is necessary that $\alpha \in [\alpha_0, +\infty)$, with

$$\alpha_0 = \max_{1 \leq i \leq n} \left((b_{ii} + \sqrt{b_{ii}^2 + 4c_{ii}})/2 \right) > 0, \quad |c_{ij}| \geq \alpha b_{ij}, \quad (i \neq j). \tag{2.4}$$

Theorem 2.1. *If the condition* (2.4) *be satisfied and* C *is a nonsingular* M-*matrix, then* R *is a nonsingular* M-*matrix and the equation* (2.1) *has a minimal nonnegative solution* S_a.

Proof. It is clear that R is a Z-matrix, by the condition (2.4). If C is a nonsingular M-matrix, then by Lemma1.1, $Cv > 0$ for some $v > 0$. Take $\delta > 0$ small enough so that $Cv - \delta(\alpha^2 I - \alpha B - C)v > 0$. It follows that

$$\begin{bmatrix} \alpha I & -I \\ -\alpha^2 I + \alpha B + C & \alpha I - B \end{bmatrix} \begin{bmatrix} v + \delta v \\ \alpha v \end{bmatrix} = \begin{bmatrix} \alpha \delta v \\ Cv - \delta(\alpha^2 I - \alpha B - C)v \end{bmatrix} > 0.$$

Hence,

$$R = \begin{bmatrix} \alpha I & -I \\ -\alpha^2 I + \alpha B + C & \alpha I - B \end{bmatrix}$$

is a nonsingular M-matrix. It follows from Lemma 1.2 that the Eq. (2.1) has a minimal nonnegative solution S_a under the condition (2.4). □

In order to find S_a, we can apply Algorithm 1.1 to the equation (2.1) with $Y_0 = 0 \in R^{n \times n}$. A specific algorithm is described as follows.

Algorithm 2.1.
1) Set $Y_0 = 0 \in R^{n \times n}$.
2) For $k = 0, 1, \cdots$, until $\{Y_k\}$ convergence

   2.1) determine $\alpha_0 = \max_{1 \leq i \leq n} \left\{ (b_{ii} + \sqrt{b_{ii}^2 + 4c_{ii}})/2 \right\} > 0$ and $\beta \geq \max \left\{ \max_{1 \leq i \leq n} (\alpha - b_{ii}), \alpha \right\}$

subject to $\alpha \in [\alpha_0, +\infty)$ and $|c_{ij}| \geq \alpha b_{ij} \quad (i \neq j)$.

2.2) compute $\{Y_k\}$ form the following matrix equations

$$\begin{cases} Y_{k+\frac{1}{2}}((\beta + \alpha)I - Y_k) = ((\beta - \alpha)I + B)Y_k + (\alpha^2 I - \alpha B - C), \\ ((\beta + \alpha)I - B - Y_{k+\frac{1}{2}})Y_{k+1} = Y_{k+\frac{1}{2}}((\beta - \alpha)I) + (\alpha^2 I - \alpha B - C). \end{cases}$$

2.3) If $\|Y_{k+1} - Y_k\| \leq \varepsilon$, stop; else set $k = k + 1$, go to 2.2).

We know from Theorem 2.1 and Theorem in [7] that the matrix sequence $\{Y_k\}$ generated by Algorithm 2.1 is convergent to S_a.

It is clear that $\alpha I - S_a$ is the solution of the quadratic matrix equation (1.1) by the relationship between the equations (1.1) and (2.1). Moreover, we can easily prove that $\alpha I - S_a$ is a nonsingular M-matrix, which means that we find a nonsingular M-matrix solution of the equation (1.1).

3. Numerical Results

In this section, Algorithm 2.1 is applied to solve the quadratic matrix equation (1.1). We give some examples to illustrate the performance of the proposed algorithm. All experiments are carried out in Matlab 6.5. We compute the number of iteration steps, the CPU time and the residual norm of the solution.

Example 3.1. The quadratic matrix equation (1.1) is defined by

$$B = - \begin{bmatrix} aI_{10} & \\ & -bI_{10} \end{bmatrix}, \quad C = - \begin{bmatrix} -1 & 0.5 & & \\ & \ddots & \ddots & \\ & & \ddots & 0.5 \\ 1 & & & -1 \end{bmatrix},$$

where $a = 1$, $b = -1$ and C is a nonsingular M-matrix. It can be verified that B, C satisfy the conditions in Theorem 2.1. Taking $Y_0 = 0$, we apply Algorithm 2.1 to compute the M-matrix solution of the quadratic matrix equation (1.1) with the parameters $\alpha = 4, \beta = 6$. The quality δ is defined as the residual norm, i.e., $\delta = \|X_k^2 - BX_k - C\|$, where X_k is the k-th approximate solution. The residual norm is shown in Table 3.1 and the CPU time $\tau = 0.015'$. For different values of α, β satisfing the conditions of Algorithm 2.1, we can obtain the numerical solution by Algorithm 2.1.

Table 3.1: The residual norm of the solution.

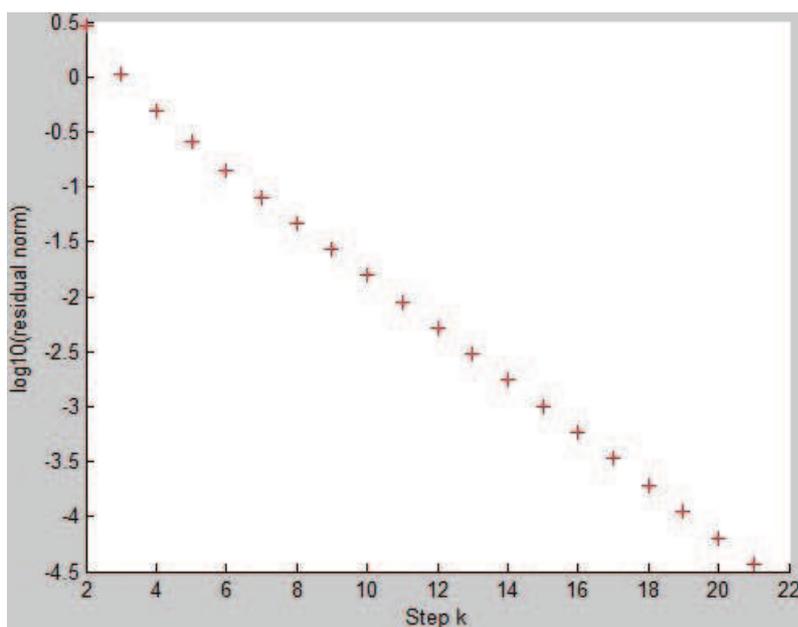| k	$\|Y_{k+1} - Y_k\|$	$\delta = \|X_k^2 - BX_k - C\|$
5	0.0961	0.2409
10	0.0602	0.0520
15	0.0046	0.0118
20	0.0011	0.0028
25	2.7668e-004	7.2203e-004
30	7.2579e-005	1.8994e-004
38	8.9059e-006	2.3369e-005

Fig. 3.1.

Example 3.2. The quadratic matrix equation (1.1) is defined by

$$
B = - \begin{bmatrix} 1 & -0.125 & & \\ & \ddots & \ddots & \\ & & \ddots & -0.125 \\ -0.25 & & & 1 \end{bmatrix}, \quad C = - \begin{bmatrix} -1 & 0.5 & & \\ & \ddots & \ddots & \\ & & \ddots & 0.5 \\ 1 & & & -1 \end{bmatrix},
$$

where $B, C \in R^{20 \times 20}$ satisfy the conditions in Theorem 2.1. We set $\alpha = 4, \beta = 6$. Algorithm 2.1 converges after 21 iterations. The residual norm is $\delta = \left\| X_k^2 - BX_k - C \right\| = 3.67 \times 10^{-5}$ and the CPU time $\tau = 0.063'$. Fig. 3.1 shows that Algorithm 2.1 is feasible and effective.

4. Conclusions

Using the alternately implicit iteration method, a new method for solving the quadratic matrix equation is developed. The corresponding theory of the method is presented. The numerical results show that the proposed method is effective.

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