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A PRIORI AND A POSTERIORI ERROR ESTIMATES OF A WEAKLY OVER-PENALIZED INTERIOR PENALTY METHOD FOR NON-SELF-ADJOINT AND INDEFINITE PROBLEMS*

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Abstract

In this paper, we study a weakly over-penalized interior penalty method for non-self-adjoint and indefinite problems. An optimal a priori error estimate in the energy norm is derived. In addition, we introduce a residual-based a posteriori error estimator, which is proved to be both reliable and efficient in the energy norm. Some numerical testes are presented to validate our theoretical analysis.

Mathematics subject classification: 65N15, 65N30.
Key words: Interior penalty method, Weakly over-penalization, Non-self-adjoint and indefinite, A priori error estimate, A posteriori error estimate.

1. Introduction

We are devoted to studying a weakly over-penalized interior penalty (WOPIP) method [7] for the following non-self-adjoint and indefinite problems

$$\begin{aligned} -\nabla\cdot(a\nabla u)+\mathbf{b}\cdot\nabla u+cu=f, &\quad \text{in}\quad \Omega,\\ u=0, &\quad \text{on}\quad \partial\Omega, \end{aligned}\tag{1.1}$$

where $\Omega\subset R^2$ is a bounded polygonal domain with boundary $\partial\Omega$. Here we assume that the data of (1.1), i.e., $\mathbf{D}=(a,\mathbf{b},c)$ satisfy the following property:

1. There exists $a_0>0$ such that $0<a_0<a$ and $c\geq 0$;
2. $a\in W^1_\infty(\Omega)$, $\mathbf{b}\in\big(L^\infty(\Omega)\big)^2$ and $c\in L^\infty(\Omega)$ with $M=\max\{\|a\|_{L^\infty(\Omega)},\|\mathbf{b}\|_{L^\infty(\Omega)},\|c\|_{L^\infty(\Omega)}\}$;
3. $f\in L^2(\Omega)$.

The WOPIP method belongs to a class of discontinuous Galerkin (DG) methods, which was first proposed in [7] by Brenner et al. to solve second order elliptic equations. DG methods

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for elliptic problems have been initially proposed in [2, 31] in the 1970s-1980s. In recent years they have gained much interest due to their suitability for hp-adaptive techniques, flexibility in handling inhomogeneous boundary conditions and curved boundaries, and their flexibility in handling highly nonuniform and unstructured meshes. The reader is referred to [14] for applications of these methods for a wide variety of problems, and to [3] for an over review of these methods for elliptic problems and their a priori error analysis. For more details of the a priori error estimates for second elliptic problems, please refer to [23]. For the theory of a posteriori error bounds for DG methods, the residual-based error estimators measured in mesh-dependent energy norms have been presented in [5, 19, 20, 22, 24], and further been studied in [1, 33]. Some other work on the a posteriori error estimates of DG methods can be found in [15, 26, 28, 29]. For the WOPIP method for second order equations, its a priori error estimate was provided in [7], where some advantages of this method were also discussed, e.g., compared with many well-known DG methods presented in [3], the WOPIP method has less computational complexity and is easy to implement. Subsequently, a residual-based posteriori error estimator was presented in [8]. More applications of the WOPIP methods are to use them to solve the biharmonic problem [9] and Stokes equations [4].

The non-self-adjoint and indefinite problems (1.1) often appear in dealing with flow in porous media. To the best of our knowledge, there exists no work on the a posteriori error estimates of DG methods for non-self-adjoint and indefinite problems. The main objective of this paper is to give a residual-based error estimator of the WOPIP DG method for (1.1). In this case, two main difficulties should be overcome, one arises from the effect of a nonsymmetric and indefinite bilinear form, the other stems from the nonconformity of the WOPIP DG method.

The rest of our paper is organized as follows. We introduce some notations and recall the WOPIP method in Section 2. An optimal a priori error estimate of the WOPIP method in the energy norm is provided in Section 3. A residual-based a posteriori error estimator of the WOPIP method is presented in Section 4. Moreover, both the upper bound and lower bound of the error estimator are proved in the energy norm. Finally, some numerical experiments which validate our theoretical results are given in Section 5.

2. Preliminaries and Notations

For a bounded domain $\mathcal{D}$ in R^2, we denote by $H^s(\mathcal{D})$ the standard Sobolev space of functions with regularity exponent $s \geq 0$, associated with norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. When $s = 0$, $H^0(\mathcal{D})$ can be written by $L^2(\mathcal{D})$. When $\mathcal{D} = \Omega$, the norm $\|\cdot\|_{s,\Omega}$ is simply written by $\|\cdot\|_s$. $H_0^s(\mathcal{D})$ is the subspace of $H^s(\mathcal{D})$ with vanishing trace on $\partial\mathcal{D}$.

Let $\mathcal{T}_h$ be a regular decompositions of Ω into triangles $\{T\}$, h_T denotes the diameter of T and $h = \max_{T\in\mathcal{T}_h} h_T$. Denote ε_h^0 by the set of interior edges of elements in $\mathcal{T}_h$, and ε_h^∂ by the set of boundary edges. Set $\varepsilon_h = \varepsilon_h^0 \cup \varepsilon_h^\partial$. The length of any edge $e \in \varepsilon_h$ is denoted by h_e. Further, we associate a fixed unit normal $\mathbf{n}$ with each edge $e \in \varepsilon_h$ such that for edges on the boundary $\partial\Omega$, $\mathbf{n}$ is the exterior unit normal.

Let e be an interior edge in ε_h^0 shared by elements T_1 and T_2. For a scalar piecewise smooth function φ, with $\varphi^i = \varphi|_{T_i}$, we define the following jump by

$$[\![\varphi]\!] = \varphi^1 - \varphi^2, \quad \text{on } e \in \varepsilon_h^0.$$

For a boundary edge $e \in \varepsilon_h^\partial$, we set

$$[\![\varphi]\!] = \varphi.$$

The weak formulation of (1.1) is to find $u \in V \triangleq H_0^1(\Omega)$ such that

$$a(u,v) = \int_\Omega f v dx, \quad \forall v \in V, \tag{2.1}$$

with

$$a(u,v) = \int_\Omega \big(a\nabla u \cdot \nabla v + (\mathbf{b}\cdot\nabla u)v + cuv\big)dx. \tag{2.2}$$

Define the discontinuous Galerkin finite element space by

$$V_h = \big\{v \in L^2(\Omega) : v|_T \in P_1(T), \quad \forall T \in \mathcal{T}_h\big\}. \tag{2.3}$$

Following [7], we present a weakly over-penalized interior penalty method for the problems (1.1): find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v dx, \quad \forall v \in V_h, \tag{2.4}$$

where

$$a_h(u_h,v) = \sum_{T\in\mathcal{T}_h}\int_T \big(a\nabla u_h \cdot \nabla v + (\mathbf{b}\cdot\nabla u_h)v + cu_h v\big)dx + \sum_{e\in\varepsilon_h} h_e^{-2}\big(\Pi_e^0[\![u_h]\!]\big)\big(\Pi_e^0[\![v]\!]\big), \tag{2.5}$$

with $\Pi_e^0 v$ defined by the mean of v over the $e \in \varepsilon_h$, i.e.,

$$\Pi_e^0 v = \frac{1}{h_e}\int_e v ds.$$

We may note that the WOPIP method above dose not have the Galerkin orthogonality, i.e.,

$$a_h(u-u_h, v) \neq 0, \quad v \in V_h.$$

Define the mesh-dependent norm $\|\!| \cdot |\!\|_h$ on $V + V_h$ by

$$\|\!|v|\!\|_h = \Big(\sum_{T\in\mathcal{T}_h}\big(\|\nabla v\|_{0,T}^2 + \|v\|_{0,T}^2\big) + \sum_{e\in\varepsilon_h} h_e^{-2}\big(\Pi_e^0[\![v]\!]\big)^2\Big)^{\frac{1}{2}}. \tag{2.6}$$

Let $V_c \subset H_0^1(\Omega)$ be the conforming P_1 finite element space associated with the triangulation $\mathcal{T}_h$. We construct an enriching operator $E : V_h \to V_c$ by average

$$(Ev)(p) = \frac{1}{|\mathcal{T}_p|}\sum_{T\in\mathcal{T}_h} v|_T(p), \tag{2.7}$$

where p is any interior node for $\mathcal{T}_h$, $\mathcal{T}_p$ is the set of all triangles sharing the node p, and $|\mathcal{T}_p|$ is the number of triangles in $\mathcal{T}_p$.

The enriching operator E above satisfies [6, 8, 18, 20]

$$\sum_{T\in\mathcal{T}_h}\Big(h_T^{-2}\|v - Ev\|_{0,T}^2 + \|\nabla(v-Ev)\|_{0,T}^2\Big) \le C\Big(\sum_{e\in\varepsilon_h} h_e^{-1}\|[\![v]\!]\|_{0,e}^2\Big), \quad \forall v \in V_h. \tag{2.8}$$

We need the following result by using Clément or Scott-Zhang interpolation [13, 27].

Lemma 2.1. *For any $\psi \in H_0^1(\Omega)$, there exists a piecewise linear approximation $(\psi_h = \Pi_h\psi) \in V_c$ such that*

$$\|\psi - \Pi_h\psi\|_{0,T} \le Ch_T\|\nabla\psi\|_{0,\tilde{T}}, \quad \forall T \in \mathcal{T}_h, \tag{2.9}$$

$$\|\psi - \Pi_h\psi\|_{0,e} \le Ch_e^{\frac{1}{2}}\|\nabla\psi\|_{0,\tilde{e}}, \quad \forall e \in \varepsilon_h, \tag{2.10}$$

where $\tilde{T}$ is the union of all elements in $\mathcal{T}_h$ having nonempty intersection with T, $\tilde{e} = \tilde{T}_1 \cup \tilde{T}_2$ with $e = T_1 \cap T_2$, and $C > 0$ is a constant depending only on minimum angle of $\mathcal{T}_h$.

3. A Priori Error Analysis

The analysis for the a priori error estimate is largely based on the reference [17]. First, we have the following lemma which can be immediately derived from Cauchy-Schwarz inequality.

Lemma 3.1. *There exists a constant $C > 0$ independent of h but depending on a_0, and M, such that*

$$|a_h(\phi, v)| \leq C\|\phi\|_h\|v\|_h, \quad \forall \phi,\ v \in V + V_h. \tag{3.1}$$

Then, we prove the Gårding-type inequality on $a_h(\cdot,\cdot)$ in the following lemma.

Lemma 3.2. *There exist two constants $C_1 > 0$ and $C_2 > 0$ independent h but depending on a_0, and M, such that*

$$a_h(v, v) \geq C_1\|v\|_h^2 - C_2\|v\|_0^2, \quad \forall v \in V + V_h. \tag{3.2}$$

Proof. By the definition of $a(\cdot,\cdot)$, we have

$$a_h(v, v) = \sum_{T\in\mathcal{T}_h}\int_T \big(a\nabla v\cdot\nabla v + (\mathbf{b}\cdot\nabla v)v + cv^2\big)dx + \sum_{e\in\varepsilon_h} h_e^{-2}\big(\Pi_e^0[\![v]\!]\big)^2. \tag{3.3}$$

By the assumptions on the data $D = (a, \mathbf{b}, c)$, and using Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T \big(a\nabla v\cdot\nabla v + (\mathbf{b}\cdot\nabla v)v + cv^2\big)dx\\
\geq& \sum_{T\in\mathcal{T}_h}\int_T \big(a|\nabla v|^2 + cv^2\big)dx - \sum_{T\in\mathcal{T}_h}\int_T |\mathbf{b}||\nabla v||v|dx\\
\geq& a_0\sum_{T\in\mathcal{T}_h}\int_T \big(|\nabla v|^2 + v^2\big)dx - a_0\|v\|_0^2 - M\Big(\sum_{T\in\mathcal{T}_h}\int_T|\nabla v|^2dx\Big)^{1/2}\|v\|_0\\
\geq& a_0\sum_{T\in\mathcal{T}_h}\big(\|\nabla v\|_{0,T}^2 + \|v\|_{0,T}^2\big)dx - \frac{\alpha}{2}\sum_{T\in\mathcal{T}_h}\|\nabla v\|_{0,T}^2 - (\frac{M^2}{2\alpha} + a_0)\|v\|_0^2.
\end{aligned} \tag{3.4}$$

Choosing α to make $a_0 - \frac{\alpha}{2} > 0$, and substituting (3.4) into (3.3), we obtain the lemma. □

Let $\mathcal{I}_h$ be the Crouzeix-Raviart interpolation operator defined in [7]. Similar to Lemma 3.3 in [7], we have the following lemma.

Lemma 3.3. *There exists a constant $C > 0$ independent of h but depending the minimum angle of $\mathcal{T}_h$, such that*

$$\|\phi - \mathcal{I}_h\phi\|_h \leq Ch\|\phi\|_2. \tag{3.5}$$

In particular, in the above lemma, if we choose ϕ be the solution of the problem (1.1), since the elliptic regularity $\|u\|_2 \leq C\|f\|_0$ holds, then we have

$$\|u - \mathcal{I}_h u\|_h \leq Ch\|u\|_2 \leq Ch\|f\|_0. \tag{3.6}$$

The following lemma will be used in the proof of the a priori error estimates.

Lemma 3.4. *Let $q \in L^2(\Omega)$, then for sufficiently small h, there exists a unique $\phi_h \in V_h$ satisfying*

$$a_h(v_h, \phi_h) = \int_\Omega q v_h dx, \quad \forall v_h \in V_h. \tag{3.7}$$

Furthermore, ϕ_h satisfy

$$|||\phi_h|||_h \leq C\|q\|_0, \tag{3.8}$$

where $C > 0$ is independent of h but depending on a_0, M and minimum angle of $\mathcal{T}_h$.

Proof. Since (3.7) leads to a system of linear algebraic equations, it is enough to prove uniqueness. Setting $v_h = \phi_h$ in (3.7) and using Lemma 3.2, we obtain

$$\begin{aligned} & C_1 |||\phi_h|||_h^2 - C_2\|\phi_h\|_0^2 \\ \leq\ & a_h(\phi_h, \phi_h) = \int_\Omega q\phi_h dx \leq \ \|q\|_0\|\phi_h\|_0. \end{aligned}$$

Therefore, we get

$$|||\phi_h|||_h \leq C\|q\|_0 + C\|\phi_h\|_0. \tag{3.9}$$

In order to estimate $\|\phi_h\|_0$ in terms of $|||\phi_h|||_h$, we apply the standard Aubin-Nitsche duality argument. For ϕ_h, we consider the following auxiliary problem

$$\begin{aligned} -\nabla \cdot (a\nabla\varphi) + \mathbf{b} \cdot \nabla\varphi + c\varphi = \phi_h, \quad & \text{in } \ \Omega, \\ \varphi = 0, \quad & \text{on } \ \partial\Omega. \end{aligned} \tag{3.10}$$

Then from the assumptions on the problem (1.1) in the introduction, we can see that φ satisfies the following elliptic regularity

$$\|\varphi\|_2 \leq C\|\phi_h\|_0. \tag{3.11}$$

Multiplying (3.10) by ϕ_h and integrating over Ω, then applying integration by parts, we obtain

$$\begin{aligned} \|\phi_h\|_0^2 &= a_h(\varphi, \phi_h) - \sum_{e\in\varepsilon_h} \int_e (a\nabla\varphi \cdot \mathbf{n})[\![\phi_h]\!] ds \\ &= a_h(\varphi - \mathcal{I}_h\varphi, \phi_h) + a_h(\mathcal{I}_h\varphi, \phi_h) - \sum_{e\in\varepsilon_h} \int_e (a\nabla\varphi \cdot \mathbf{n})[\![\phi_h]\!] ds. \end{aligned} \tag{3.12}$$

For the first term in the above equality, using Lemma 3.3 and (3.11) we have

$$a_h(\varphi - \mathcal{I}_h\varphi, \phi_h) \leq C|||\varphi - \mathcal{I}_h\varphi|||_h |||\phi_h|||_h \leq Ch\|\varphi\|_2 |||\phi_h|||_h \leq Ch\|\phi_h\|_0 |||\phi_h|||_h. \tag{3.13}$$

For the second term, in view of (3.7), and using the stability of interpolation $\mathcal{I}_h$ in $H^2(\Omega)$ [7], we get

$$\begin{aligned} a_h(\mathcal{I}_h\varphi, \phi_h) &= \int_\Omega q\mathcal{I}_h\varphi dx \leq \|q\|_0\|\mathcal{I}_h\varphi\|_0 \\ &\leq \|q\|_0\|\mathcal{I}_h\varphi\|_2 \leq C\|q\|_0\|\varphi\|_2. \end{aligned} \tag{3.14}$$

For the third term, recalling the result in Lemma 3.2 in [7], we have

$$\sum_{e\in\varepsilon_h} \int_e (a\nabla\varphi \cdot \mathbf{n})[\![\phi_h]\!] ds \leq C\Big(\inf_{\varphi_h\in V_h} |||\varphi - \varphi_h|||_h + h\|\phi_h\|_0\Big) |||\phi_h|||_h. \tag{3.15}$$

Setting $\varphi_h = \mathcal{I}_h\varphi$ in the above equality and using Lemma 3.3 and (3.11), we get

$$\sum_{e\in\varepsilon_h}\int_e (a\nabla\varphi\cdot\mathbf{n})[\![\phi_h]\!]ds \le Ch\|\phi_h\|_0 |\!|\!|\phi_h|\!|\!|_h. \tag{3.16}$$

From (3.13), (3.14), (3.16), and using the elliptic regularity (3.11), we obtain

$$\|\phi_h\|_0 \le Ch|\!|\!|\phi_h|\!|\!|_h + \|q\|_0. \tag{3.17}$$

Substituting (3.17) into (3.9), we get the the estimate (3.8) for sufficiently small h. Moreover, (3.8) implies a unique solution of (3.7), thus the proof is completed. □

Based on the above lemmas, we formulate the main result of this section in the following theorem.

Theorem 3.1. *Let u be the solution of the problem* (1.1)*, and u_h be the numerical solution of the WOPIP method in* (2.4)*. Then, for sufficiently small h, there exists a constant $C > 0$ independent of h but depending on a_0, M and minimum angle of $\mathcal{T}_h$, such that*

$$|\!|\!|u - u_h|\!|\!|_h \le Ch\|f\|_0. \tag{3.18}$$

Proof. Let $e_r = u - u_h$ be split into $e_r = \xi + \chi$, where $\xi = u - \mathcal{I}_h u$ and $\chi = \mathcal{I}_h u - u_h$. Using lemmas 3.1 and 3.2, we have

$$\begin{aligned} C_1|\!|\!|\chi|\!|\!|_h^2 - C_2\|\chi\|_0^2 \le a_h(\chi,\chi) &= a_h(\mathcal{I}_h u - u, \chi) + a_h(u - u_h, \chi) \\ &\le C|\!|\!|\xi|\!|\!|_h |\!|\!|\chi|\!|\!|_h + a_h(u - u_h, \chi). \end{aligned}$$

For the term $a_h(u - u_h, \chi)$ in the above inequality, using Lemma 3.2 in [7] we obtain

$$a_h(u - u_h, \chi) \le C\Big(\inf_{v_h\in V_h} |\!|\!|u - v_h|\!|\!|_h + h\|f\|_0\Big)|\!|\!|\chi|\!|\!|_h. \tag{3.19}$$

Noting that $\|\chi\|_0 \le |\!|\!|\chi|\!|\!|_h$, then we have

$$|\!|\!|\chi|\!|\!|_h \le C|\!|\!|\xi|\!|\!|_h + C\|\chi\|_0 + C\Big(\inf_{v_h\in V_h} |\!|\!|u - v_h|\!|\!|_h + h\|f\|_0\Big). \tag{3.20}$$

In order to estimate $\|\chi\|_0$, we set $q = \chi$ and $v_h = \chi$ in Lemma 3.4. Using Lemma 3.1 and (3.19), we have

$$\begin{aligned} \|\chi\|_0^2 = a_h(\chi, \phi_h) &= a_h(\mathcal{I}_h u - u_h, \phi_h) \\ &= a_h(\mathcal{I}_h u - u, \phi_h) + a_h(u - u_h, \phi_h) \\ &\le |\!|\!|\xi|\!|\!|_h |\!|\!|\phi_h|\!|\!|_h + C\Big(\inf_{v_h\in V_h} |\!|\!|u - v_h|\!|\!|_h + h\|f\|_0\Big)|\!|\!|\phi_h|\!|\!|_h. \end{aligned}$$

Using (3.8) in Lemma 3.4, we get $|\!|\!|\phi_h|\!|\!|_h \le C\|\chi\|_0$, then we have

$$\|\chi\|_0 \le C|\!|\!|\xi|\!|\!|_h + C\Big(\inf_{v_h\in V_h} |\!|\!|u - v_h|\!|\!|_h + h\|f\|_0\Big). \tag{3.21}$$

In view of (3.20) and (3.21), we obtain

$$|\!|\!|\chi|\!|\!|_h \le C|\!|\!|\xi|\!|\!|_h + C\Big(\inf_{v_h\in V_h} |\!|\!|u - v_h|\!|\!|_h + h\|f\|_0\Big). \tag{3.22}$$

Setting $v_h = \mathcal{I}_h u$ in (3.22), using (3.6) and triangle inequality, we obtain the theorem. □

Furthermore, by similar dual arguments used in [25], we can obtain the a priori error estimate in L^2-norm in the following theorem.

Theorem 3.2. *Let u be the solution of the problem* (1.1)*, and u_h be the numerical solution of the WOPIP method in* (2.4)*. Then, for sufficiently small h, there exists a constant $C > 0$ independent of h but depending on a_0, M and minimum angle of $\mathcal{T}_h$, such that*

$$\|u - u_h\|_0 \le Ch^2\|f\|_0. \tag{3.23}$$

Using Lemma 3.4, we can prove the existece of a unique solution to the problem (2.4). Let us assume that u_h^1 and u_h^2 are two distinct solutions of (2.4) and define $\theta = u_h^1 - u_h^2$. Since $a_h(\theta, v_h) = 0$ for all $v_h \in V_h$, setting $q = \theta$, $v_h = \theta$ in (3.7), we get

$$\|\theta\|_0^2 = a_h(\theta, \phi_h) = a_h(u_h^1 - u_h^2, \phi_h) = 0.$$

Then we have $\theta = 0$, i.e., $u_h^1 = u_h^2$, which leads to a contradiction. Therefore, there exists a unique solution u_h for the problem (2.4). Since the problem is finite dimensional, uniquess implies the existence of u_h.

4. A Posteriori Error Analysis

We first introduce our residual-based error estimator as follows:

1. For any $T \in \mathcal{T}_h$ we define the element residual η_T by

$$\eta_T = h_T\|\bar{f} + \nabla \cdot (a\nabla u_h) - \mathbf{b} \cdot \nabla u_h - cu_h\|_{0,T}, \tag{4.1}$$

where $\bar{f}$ is the piecewise constant function which takes the mean value of f on $T \in \mathcal{T}_h$

$$\bar{f}|_T = \frac{1}{|T|}\int_T f dx, \quad \forall T \in \mathcal{T}_h.$$

2. For any $e \in \varepsilon_h$, we define the jump residual $\eta_{e,1}$ by

$$\eta_{e,1}^2 = h_e^{-2}|\Pi_e^0[\![u_h]\!]|^2 + h_e^{-1}\|[\![u_h]\!]\|_{0,e}^2. \tag{4.2}$$

3. For any $e \in \varepsilon_h^0$, we define the jump residual $\eta_{e,2}$ by

$$\eta_{e,2}^2 = h_e\|[\![(a\nabla u_h) \cdot \mathbf{n}]\!]\|_{0,e}^2. \tag{4.3}$$

Then, the error estimator η_h is defined by

$$\eta_h^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{e\in\varepsilon_h} \eta_{e,1}^2 + \sum_{e\in\varepsilon_h^0} \eta_{e,2}^2. \tag{4.4}$$

4.1. Reliability

In this subsection, we shall prove the reliability of the error estimator η_h.

Theorem 4.1. *Let u denote the solution of the problem* (1.1)*, and u_h denote the numerical solution of the WOPIP method in* (2.4)*. Then for sufficiently small h, there exist constants $C_R > 0, C_P > 0$ depending on a_0, M and the minimum angle of $\mathcal{T}_h$ such that*

$$|\!|\!|u - u_h|\!|\!|_h \le C_R\eta_h + C_P\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|f - \bar{f}\|_{0,T}^2\Big)^{\frac{1}{2}}. \tag{4.5}$$

Proof. Following [16,20], we set $e_r = u-u_h = e_c+e_d$, with $e_c = u-Eu_h$ and $e_d = Eu_h-u_h$, here E is the enriching operator defined in the section 2. Note that the terms $e_c = u - Eu_h$ and $e_d = Eu_h - u_h$ are referred as conforming error and nonconforming error. By the triangle inequality, we get

$$\|e_r\|_h \le \|e_c\|_h + \|e_d\|_h. \tag{4.6}$$

First, we bound the second term $\|e_d\|_h$ on the right-hand side of the above inequality. Since $\Pi_e^0[\![Eu_h]\!] = 0$, by the property of enriching operator E in (2.8), the second term $\|e_d\|_h$ can be bounded by

$$\begin{aligned}\|e_d\|_h^2 &= \sum_{T\in\mathcal{T}_h}\int_T \left(\|\nabla(Eu_h-u_h)\|_{0,T}^2 + \|Eu_h-u_h\|_{0,T}^2\right) + \sum_{e\in\varepsilon_h} h_e^{-2}\left(\Pi_e^0[\![u_h]\!]\right)^2\\ &\le C\Big(\sum_{e\in\varepsilon_h} h_e^{-1}\|[\![u_h]\!]\|_{0,e}^2\Big) + \sum_{e\in\varepsilon_h} h_e^{-2}\left(\Pi_e^0[\![u_h]\!]\right)^2\\ &\le C\sum_{e\in\varepsilon_h}\eta_{e,1}^2. \end{aligned}\tag{4.7}$$

Then it leaves us to bound the first term $\|e_c\|_h$ on the right-hand side of (4.6). Let Π_h denote the Clément or Scott-Zhang interpolation in Lemma 2.1, then $\Pi_h e_c \in V_c$ and we define $\zeta = e_c - \Pi_h e_c$. Denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\Omega)$, then $\langle f, e_c\rangle = \int_\Omega f e_c dx$, thus $a_h(u, e_c) = \langle f, e_c\rangle$, we then have

$$\begin{aligned} a_h(e_r, e_c) &= a_h(u, e_c) - a_h(u_h, e_c)\\ &= \langle f, e_c\rangle - a_h(u_h,\zeta) - a_h(u_h, \Pi_h e_c)\\ &= \langle f,\zeta\rangle - a_h(u_h,\zeta), \end{aligned}$$

which implies

$$a_h(e_c, e_c) = \langle f,\zeta\rangle - a_h(u_h,\zeta) - a_h(e_d, e_c). \tag{4.8}$$

By the definition of $a_h(\cdot,\cdot)$, integrating by parts, and using Cauchy-Schwarz inequality, (2.9) and (2.10) in Lemma 2.1, we obtain

$$\begin{aligned} &\langle f,\zeta\rangle - a_h(u_h,\zeta)\\ =&\sum_{T\in\mathcal{T}_h}\int_T\Big(\bar f + \nabla\cdot(a\nabla u_h) - \mathbf{b}\cdot\nabla u_h - cu_h\Big)\zeta dx + \sum_{T\in\mathcal{T}_h}\int_T (f-\bar f)\zeta dx\\ &- \sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(a\nabla u_h\cdot\mathbf{n}\right)\zeta ds + \sum_{e\in\varepsilon_h} h_e^{-2}\left(\Pi_e^0[\![u_h]\!]\right)\left(\Pi_e^0[\![\zeta]\!]\right)\\ \le&\sum_{T\in\mathcal{T}_h}(h_T\|\bar f + \nabla\cdot(a\nabla u_h) - \mathbf{b}\cdot\nabla u_h - cu_h\|_{0,T})(h_T^{-1}\|\zeta\|_{0,T})\\ &+\sum_{T\in\mathcal{T}_h}(h_T\|f-\bar f\|_{0,T})(h_T^{-1}\|\zeta\|_{0,T}) + \sum_{e\in\varepsilon_h^0}(h_e^{\frac12}\|[\![(a\nabla u_h)\cdot\mathbf{n}]\!]\|_{0,e})(h_e^{-\frac12}\|\zeta\|_{0,e})\\ &+\sum_{e\in\varepsilon_h} h_e^{-2}(\Pi_e^0[\![u_h]\!])(\Pi_e^0[\![\zeta]\!]) \end{aligned}$$

$$
\begin{aligned}
&\leq C\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\bar{f}+\nabla\cdot(a\nabla u_h)-\mathbf{b}\cdot\nabla u_h-cu_h\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla e_c\|_{0,T}^2\Big)^{1/2}\\
&\quad+C\Big(\sum_{e\in\varepsilon_h^0} h_e\|[\![(a\nabla u_h)\cdot\mathbf{n}]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla e_c\|_{0,T}^2\Big)^{1/2}\\
&\quad+C\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|f-\bar{f}\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla e_c\|_{0,T}^2\Big)^{1/2}\\
&\leq C\Big(\big(\sum_{T\in\mathcal{T}_h}\eta_T^2\big)^{1/2}+\big(\sum_{e\in\varepsilon_h^0}\eta_{e,2}^2\big)^{1/2}+\big(\sum_{T\in\mathcal{T}_h}h_T^2\|f-\bar{f}\|_{0,T}^2\big)^{1/2}\Big)\|\!|e_c|\!\|_h. \qquad (4.9)
\end{aligned}
$$

We note that in the last step of the above inequality we use the fact $\Pi_e^0[\![\zeta]\!]=0$ on ε_h.

On the other hand, using Cauchy-Schwarz inequality and the property of enriching operator E in (2.8), and noting that $\Pi_e^0[\![e_c]\!]=0$, we have

$$
\begin{aligned}
a_h(e_d,e_c)&=\sum_{T\in\mathcal{T}_h}\int_T a\nabla e_d\nabla e_c+(\mathbf{b}\cdot\nabla e_d)e_c+ce_de_c dx\\
&\leq C\sum_{T\in\mathcal{T}_h}\Big(\|\nabla e_d\|_{0,T}\|\nabla e_c\|_{0,T}+\|\nabla e_d\|_{0,T}\|e_c\|_{0,T}+\|e_d\|_{0,T}\|e_c\|_{0,T}\Big)\\
&\leq C\Big(\sum_{e\in\varepsilon_h}h_e^{-1}\|[\![u_h]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla e_c\|_{0,T}^2\Big)^{1/2}\\
&\quad+C\Big(\sum_{e\in\varepsilon_h}h_e^{-1}\|[\![u_h]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|e_c\|_{0,T}^2\Big)^{1/2}\\
&\quad+C\Big(\sum_{e\in\varepsilon_h}h_e\|[\![u_h]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|e_c\|_{0,T}^2\Big)^{1/2}\\
&\leq C\Big(\sum_{e\in\varepsilon_h}\eta_{e,1}^2\Big)^{1/2}\|\!|e_c|\!\|_h. \qquad (4.10)
\end{aligned}
$$

From the Gårding type inequality in Lemma 3.2, we obtain

$$
C_1\|\!|e_c|\!\|_h^2\leq a_h(e_c,e_c)+C_2\|e_c\|_0^2. \qquad (4.11)
$$

Moreover, using the technique in [12, 25], we have the following estimate: for any $\epsilon>0$ the exists a $\epsilon_0(\epsilon)$ such that for the meshsize $h\in(0,\epsilon_0]$

$$
\|e_c\|_0\leq\epsilon\|\!|e_c|\!\|_h. \qquad (4.12)
$$

Combining (4.8)–(4.12) with Cauchy-Schwarz inequality, we get

$$
\|\!|e_c|\!\|_h\leq C_1\eta_h+C_2\Big(\sum_{T\in\mathcal{T}_h}h_T^2\|f-\bar{f}\|_{0,T}^2\Big)^{1/2}. \qquad (4.13)
$$

Then the theorem follows from (4.6), (4.7) and (4.13). □

4.2. Efficiency

In this subsection, we shall prove the efficiency of the error estimator. To obtain the efficiency bound, we make use of bubble function technique introduced by Verfürth [30]. Denote by b_T

the standard polynomial bubble function on element T, and by b_e the standard polynomial bubble function on an interior edge e, shared by two elements T and T'. Then we have the following results [30, 32].

Lemma 4.1. *For any polynomial function v, there exists a constant $C > 0$ depending on the minimum angle of $\mathcal{T}_h$ such that*

$$\|b_T v\|_{0,T} \le C\|v\|_{0,T}, \tag{4.14}$$

$$\|v\|_{0,T} \le C\|b_T^{\frac{1}{2}} v\|_{0,T}, \tag{4.15}$$

$$\|\nabla(b_T v)\|_{0,T} \le C h_T^{-1}\|v\|_{0,T}. \tag{4.16}$$

Similarly, for any polynomial function w on interior edge e, there exists a constant $C > 0$ depending on the minimum angle of $\mathcal{T}_h$ such that

$$\|w\|_{0,e} \le C\|b_e^{\frac{1}{2}} w\|_{0,e}. \tag{4.17}$$

Furthermore, there exists an extension $W_b \in H_0^1(\bar{T} \cup \bar{T}')$ of $b_e w$ such that $W_b|_e = b_e w$ and

$$\|W_b\|_{0,T} \le C h_e^{\frac{1}{2}}\|w\|_{0,e}, \tag{4.18}$$

$$\|\nabla W_b\|_{0,T} \le C h_e^{-\frac{1}{2}}\|w\|_{0,e}, \tag{4.19}$$

where $C > 0$ is a constant depending on the minimum angle of $\mathcal{T}_h$.

To begin, we prove the following local bounds.

Lemma 4.2. *Let u be the solution of the problem (1.1), and u_h be the numerical solution of the WOPIP method in (2.4). Then the following local bounds hold:*

(i) *For any $T \in \mathcal{T}_h$, we have*

$$\eta_T \le C\big(\|\nabla(u-u_h)\|_{0,T} + h_T\|\nabla(u-u_h)\|_{0,T} + h_T\|u-u_h\|_{0,T} + h_T\|f-\bar{f}\|_{0,T}\big). \tag{4.20}$$

(ii) *For any interior edge $e \in \varepsilon_h^0$ which belongs to two elements T and T', we have*

$$\begin{aligned}\eta_{e,2} \le C\sum_{T\in U_e}\Big(&\|\nabla(u-u_h)\|_{0,T} + h_T\|\nabla(u-u_h)\|_{0,T}\\ &+ h_T\|u-u_h\|_{0,T} + h_T\|f-\bar{f}\|_{0,T}\Big)\end{aligned} \tag{4.21}$$

with $U_e = \{T, T'\}$.

(iii) *For any edge $e \in \varepsilon_h$, we have*

$$h_e^{-2}|\Pi_e^0[\![u_h]\!]|^2 = h_e^{-2}|\Pi_e^0[\![u-u_h]\!]|^2, \tag{4.22}$$

$$h_e^{-1}\|[\![u_h]\!]\|_{0,e}^2 = h_e^{-1}\|[\![u-u_h]\!]\|_{0,e}^2. \tag{4.23}$$

All the constants $C > 0$ appear in the above inequalities depend on a_0, M and the minimum angle of $\mathcal{T}_h$, and $\bar{f}$ is the piecewise constant function which takes the mean value of f on $T \in \mathcal{T}_h$

$$\bar{f}|_T = \frac{1}{|T|}\int_T f dx, \quad \forall T \in \mathcal{T}_h.$$

Proof. (i) Set $v_h = \bar{f} + \nabla\cdot(a\nabla u_h) - \mathbf{b}\cdot\nabla u_h - cu_h$, and $v_b = b_T v_h$. Since $-\nabla\cdot(a\nabla u) + \mathbf{b}\cdot\nabla u + cu = f$ in $L^2(T)$, we have

$$\begin{aligned}\|b_T^{\frac{1}{2}} v_h\|_{0,T}^2 &= \int_T \big(\bar{f} + \nabla\cdot(a\nabla u_h) - \mathbf{b}\cdot\nabla u_h - cu_h\big) v_b dx \\ &= \int_T (f + \nabla\cdot(a\nabla u_h) - \mathbf{b}\cdot\nabla u_h - cu_h) v_b dx + \int_T (\bar{f} - f) v_b dx \\ &= \int_T \Big(-\nabla\cdot\big(a\nabla(u-u_h)\big) + \mathbf{b}\cdot\nabla(u-u_h) + c(u-u_h)\Big) v_b dx + \int_T (\bar{f}-f) v_b dx \\ &= \int_T a\nabla(u-u_h)\nabla v_b dx + \int_T \mathbf{b}\cdot\nabla(u-u_h) v_b dx + \int_T c(u-u_h) v_b dx \\ &\quad + \int_T (\bar{f}-f) v_b dx,\end{aligned}$$

where in the last step we have used integration by parts and the fact that $v_b = 0$ on ∂T. Then by Cauchy-Schwarz inequality we have

$$\begin{aligned}\|v_h\|_{0,T}^2 \le C\Big(&\|\nabla(u-u_h)\|_{0,T}\|\nabla v_b\|_{0,T} + \|\nabla(u-u_h)\|_{0,T}\|v_b\|_{0,T} \\ &+ \|u-u_h\|_{0,T}\|v_b\|_{0,T} + \|f-\bar{f}\|_{0,T}\|v_b\|_{0,T}\Big).\end{aligned}$$

Moreover, using (4.14) and (4.16), we obtain

$$\|v_h\|_{0,T} \le C\Big(h_T^{-1}\|\nabla(u-u_h)\|_{0,T} + \|\nabla(u-u_h)\|_{0,T} + \|u-u_h\|_{0,T} + \|f-\bar{f}\|_{0,T}\Big).$$

Noting that $\eta_T = h_T\|v_h\|_{0,T}$, the above inequality gives (i).

(ii) For any interior edge $e \in \varepsilon_h^0$, set $w_h = [\![(a\nabla u_h)\cdot\mathbf{n}]\!]$, $w_b = b_e w_h$. Defining $W_b \in H_0^1(\bar{T}\cup\bar{T}')$ by the the extension of w_b which satisfies (4.18) and (4.19). Using the fact that $[\![(a\nabla u)\cdot\mathbf{n}]\!] = 0$, we get

$$\begin{aligned}\|b_e^{\frac{1}{2}} w_h\|_{0,e}^2 &= \int_e [\![(a\nabla u_h)\cdot\mathbf{n}]\!] w_b ds = \int_e [\![\big(a\nabla(u_h-u)\big)\cdot\mathbf{n}]\!] w_b ds \\ &= \sum_{T\in U_e}\left(\int_T \Big(\nabla\cdot\big(a\nabla(u_h-u)\big)\Big) W_b dx + \int_T a\nabla(u_h-u)\nabla W_b dx\right) \\ &= \sum_{T\in U_e}\int_T \Big((\bar{f}-f) + \nabla\cdot\big(a\nabla(u_h-u)\big) - \mathbf{b}\cdot\nabla(u_h-u) - c(u_h-u)\Big) W_b dx \\ &\qquad + \int_T a\nabla(u_h-u)\nabla W_b dx - \int_T (\bar{f}-f) W_b dx \\ &\qquad + \int_T \mathbf{b}\cdot\nabla(u_h-u) W_b dx + \int_T c(u_h-u) W_b dx.\end{aligned}$$

Since $-\nabla\cdot(a\nabla u) + \mathbf{b}\cdot\nabla u + cu = f$ in $L^2(T)$, in view of (4.18) and (4.19), we have

$$\begin{aligned}\|w_h\|_{0,e} \le C\sum_{T\in U_e}\Big(&h_e^{\frac{1}{2}}\|\bar{f} + \nabla\cdot(a\nabla u_h) - \mathbf{b}\cdot\nabla u_h - cu_h\|_{0,T} + h_e^{-\frac{1}{2}}\|\nabla(u-u_h)\|_{0,T} \\ &+ h_e^{\frac{1}{2}}\|f-\bar{f}\|_{0,T} + h_e^{\frac{1}{2}}\|\nabla(u-u_h)\|_{0,T} + h_e^{\frac{1}{2}}\|(u-u_h)\|_{0,T}\Big).\end{aligned}$$

Making use of the bound for η_T and the shape-regularity of the mesh, we obtain

$$\begin{aligned}h_e^{\frac{1}{2}}\|[\![(a\nabla u_h)\cdot\mathbf{n}]\!]\|_{0,e} \le C\sum_{T\in U_e}\Big(&\|\nabla(u-u_h)\|_{0,T} + h_T\|\nabla(u-u_h)\|_{0,T} \\ &+ h_T\|u-u_h\|_{0,T} + h_T\|f-\bar{f}\|_{0,T}\Big),\end{aligned}$$

which yields (ii).

(iii) Since $\Pi_e^0[\![u]\!] = 0$ on interior edges and $u = 0$ on the boundary edges, we can obtain (4.22)-(4.23) immediately. □

We further recall a relation between the jumps across edges and the norm $|\!|\!| \cdot |\!|\!|_h$ from Lemma 3.1 in [7]

$$\sum_{e\in\varepsilon_h} h_e^{-1}\|[\![v]\!]\|_{0,e}^2 \le C|\!|\!|v|\!|\!|_h, \quad \forall v \in V + V_h. \tag{4.24}$$

Based on the above lemma and (4.24), we can obtain the main result of this section in the following theorem.

Theorem 4.2. *Let u denote the solution of the problem* (1.1)*, and u_h denote the numerical solution of the WOPIP method in* (2.4)*. Then there exists a constant $C_E > 0$ depending on a_0, M and the minimum angle of $\mathcal{T}_h$ such that*

$$\eta_h \le C_E\Big(|\!|\!|u - u_h|\!|\!|_h^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|f - \bar{f}\|_{0,T}^2\Big)^{\frac{1}{2}}. \tag{4.25}$$

5. Numerical Experiments

All the numerical experiments in this section are implemented by MATLAB. In each adaptive finite element procedure, we refine the marked triangles by the bisection algorithm, which derives from the $AFEM@matlab$ implementation [11]. First, by choosing enough smooth exact solution u in the following example, we provide some results of the a priori error.

Example 5.1. We set the exact solution $u = x(1-x)y(1-y)$ with the corresponding right-hand side function f and $\Omega = (0,1)\times(0,1)$ in problems (1.1), here the data $D = (a, \mathbf{b}, c)$ is chosen such that $a = 1$, $\mathbf{b} = (1,1)$ and $c = 1$, respectively.

For this test, in Fig. 5.1, we show the energy errors $|\!|\!|u - u_h|\!|\!|_h$ with respect to the mesh size h in the logarithmic scale. The order of convergence rate which is also the absolute value of the slope of line is 1.0262, these results confirm Theorem 3.1. Moreover, in Fig. 5.2 we describe the error between the exact solution u and its numerical solution u_h.

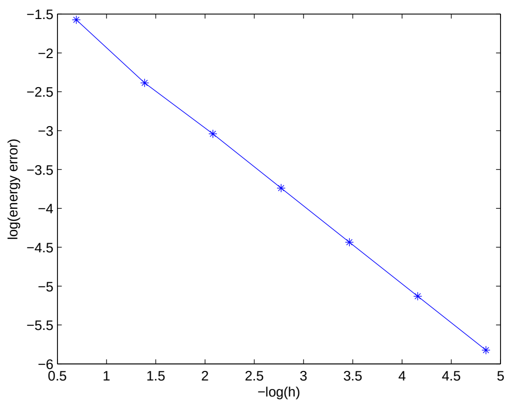

Fig. 5.1. The convergence rate for the WOPIP method.

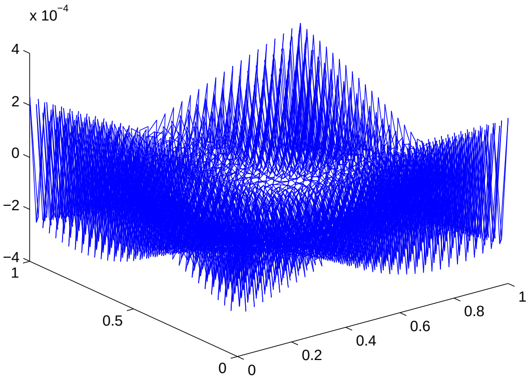

Fig. 5.2. The error between the exact solution u and its numerical solution u_h with $h = \frac{1}{64}$.

As for the a posteriori error estimates, we present some results by introducing the following L-shape domain example.

Example 5.2. We consider the problem of (1.1) with the exact solution given by $u = r^{\frac{2}{3}}\sin\left(\frac{2}{3}\theta\right)$ (in cylindrical coordinates) defined on the L-shaped domain $\Omega = (-1,1)^2 \backslash ([0,1]\times[-1,0])$, here the data $D = (a, \mathbf{b}, c)$ is chosen such that $a = 1$, $\mathbf{b} = (r\sin\theta, r\cos\theta)$ and $c = r^{1/2}$, respectively.

First, in Fig. 5.3, in log-log coordinates, we show the true error $\|u - u_h\|_h$ and the error estimator

$$\eta_h = \left(\sum_{T\in\mathcal{T}_h} \eta_T^2 + + \sum_{e\in\varepsilon_h} \eta_{e,1}^2 + \sum_{e\in\varepsilon_h^0} \eta_{e,2}^2 \right)^{1/2},$$

which are computed on a sequence of adaptive meshes as functions of number of degrees of freedom. These results validate the theoretical analysis in the Theorem 4.1 and Theorem 4.2. In Fig. 5.4, we also show the adaptive mesh of 22 level in the computational procedure. From the convergence history in Fig. 5.3, we observe the quasi-optimality of the adaptive algorithm in the sense that $\|u - u_h\|_h \approx CN^{-1/2}$ asymptotically, here N is the number of degrees of

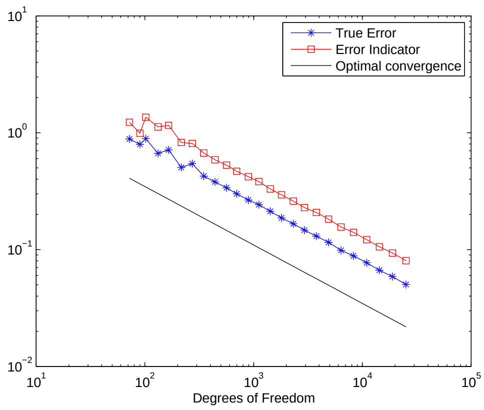

Fig. 5.3. Convergence history of the adaptive algorithm for Example 5.2.

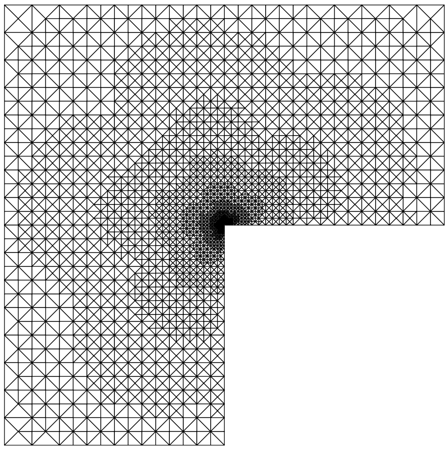

Fig. 5.4. Adaptive mesh of level 22 for Example 5.2.

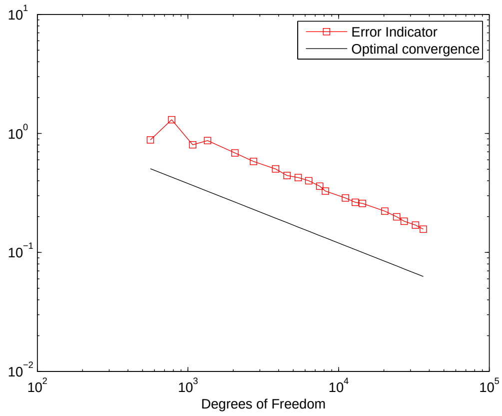

Fig. 5.5. Convergence history of the adaptive algorithm for Example 5.3.

freedom.

Example 5.3. We consider a convection-dominated diffusion problem of (1.1) on the domain $\Omega = (0,1)^2$(cf. experiment 2 in [21] and example 7.2 in [10]), the coefficients are given by

$$a = \epsilon I, \quad \epsilon = 10^{-3}, \quad \mathbf{b} = (y, 0.7 - x), \quad c = f = 0,$$

and the boudary condtions are Dirichlet type, i.e., $u = g$ on $\partial\Omega$. The data g is given by

$$g(x,y) = \begin{cases} 1, & \{0.4 + \tau \leq x \leq 0.7 - \tau, y = 0\}, \\ 0, & \partial\Omega \backslash \{0.4 \leq x \leq 0.7, y = 0\}, \\ \text{linear}, & \{0.4 \leq x \leq 0.4 + \tau, y = 0\} \text{ or } \{0.7 - \tau \leq x \leq 0.7, y = 0\}. \end{cases} \tag{5.1}$$

We set the parameter $\tau = 0.003$. The convergence history showed in Fig. 5.5 also illustrates the optimal convergence of the adaptive algorithm when the mesh size is small enough.

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References

[1] M. Ainsworth, A posteriori error estimation for discontinuous Galerkin finite element approximation, *SIAM J. Numer. Anal.*, **45** (2007), 1777-1798.

[2] D.N. Arnold, An interior penalty finite element method with discontinuous elements, *SIAM J. Numer. Anal.*, **19** (1982), 742-760.

[3] D.N. Arnold, F. Brezzi, B. Cockburn and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.*, **39** (2002), 1749-1779.

[4] A.T. Barker and S.C. Brenner, A mixed finite element method for the Stokes equations based on a weakly over-penalized symmetric interior penalty approach, *J. Sci. Comput.*, **58** (2014), 290-307.

[5] R. Becker, P. Hansbo and M. Larson, Energy norm a posteriori error estimation for discontinuous Galerkin methods, Comput. *Methods Appl. Mech. Engrg.*, **192** (2003), 723-733.

[6] S.C. Brenner, Poincaré-Friedrichs inequalities for piecewise H^1 functions, *SIAM J. Numer. Anal.*, **41** (2003), 306-324.

[7] S.C. Brenner, L. Owens and L.Y Sung, A weakly over-penalized symmetric interior penalty method, *Electron. Trans. Numer. Anal.*, **30** (2008), 107-127.

[8] S.C. Brenner, T. Gudi and L.Y. Sung, A posteriori error control for a weakly over-penalized symmetric interior penalty method, *J. Sci. Comput.*, **40** (2009), 37-50.

[9] S.C. Brenner, T. Gudi and L.Y. Sung, A weakly over-penalized symmetric interior penalty method for the biharmonic problem, *Electron. Trans. Numer. Anal.*, **37** (2010), 214-238.

[10] H. Chen, X. Xu and R.H.W. Hoppe, Convergence and quasi-optimality of adaptive nonconforming finite element methods for some nonsymmetric and indefinite problems, *Numer. Math.*, **116**(2010), 383-419.

[11] L. Chen and C. Zhang, AFEM@matlab: a Matlab package of adaptive finite element methods, Technical report, University of Maryland at College Park, 2006.

[12] Z. Chen, D.Y. Kwak and Y.J. Yon, Multigrid algorithms for nonconforming and mixed methods for nonsymmetric and indefinite problems. *SIAM J. Sci. Comput.*, **19** (1998), 502-515.

[13] P. Clément, Approximation by finite element functions using local regularization, *RAIRO Anal. Numér.*, **2** (1975), 77-84.

[14] B. Cockburn, G.E. Karniadakis and C.W. Shu (Eds.), Discontinuous Galerkin Methods–Theory, Computation and Applications, Lecture Notes in Computational Science and Engineering 11. Springer-Verlag, New York, 2000.

[15] A. Ern and J. Proft, A posteriori discontinuous Galerkin error estimates for transient convection-diffusion equations, *Appl. Math. Lett.*, **18** (2005), 833-841.

[16] E.H. Georgoulis, Discontinuous Galerkin Methods for Linear Problems: An Introduction, In E. H. Georgoulis, A. Iske, and J. Levesley (eds.), Approximation Algorithms for Complex Systems, Springer Proceedings in Mathematics, Vol. 3, Springer-Verlag, Berlin, 2011.

[17] T. Gudi and A.K. Pani, Discontinuous Galerkin methods for quasi-linear elliptic problems of nonmonotone type, *SIAM J. Numer. Anal.*, **45** (2007), 163-192.

[18] T. Gudi, Some nonstandard error analysis of discontinuous Galerkin methods for elliptic problems, *Calcolo*, **47** (2010), 239-261.

[19] P. Houston, D. Schötzau and T.P. Wihler, Energy norm a posteriori error estimation of hp-adaptive discontinuous Galerkin methods for elliptic problems, *Math. Models Methods Appl. Sci.*, **17** (2007), 33-62.

[20] O.A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, *SIAM J. Numer. Anal.*, **41** (2003), 2374-2399.

[21] K. Mekchay and R.H. Nochetto, Convergence of adaptive finite element methods for general second order linear elliptic PDEs, *SIAM J. Numer. Anal.*, **43** (2005), 1803-1827.

[22] B. Rivière and M.F. Wheeler, A posteriori error estimates for a discontinuous Galerkin method applied to elliptic problems, *Comput. Math. Appl.*, **46** (2003), 141-163.

[23] B. Rivière, M.F. Wheeler and V. Girault, A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems, *SIAM J. Numer. Anal.*, **39** (2001), 902-931.

[24] A. Romkes, S. Prudhomme and J.T. Oden, A posteriori error estimation for a new stabilized discontinuous Galerkin method, *Appl. Math. Lett.*, **16** (2003), 447-452.

[25] A.H. Schatz and J. Wang, Some new estimates for Ritz-Galerkin methods with minimal regularity assumptions, *Math. Comp.*, **65** (1996), 19-27.

[26] R. Schneider, Y. Xu and A. Zhou, An analysis of discontinuous Galerkin methods for elliptic problems, *Adv. Comput. Math.*, **25** (2006), 259-286.

[27] L.R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, *Math. Comp.*, **54** (1994), 483-493.

[28] S. Sun and M.F. Wheeler, $L^2(H^1)$-norm a posteriori error estimation for discontinuous Galerkin approximations of reactive transport problems, *J. Sci. Comput.*, **22/23** (2005), 501-530.

[29] S.K. Tomar and S.I. Repin, Efficient computable error bounds for discontinuous Galerkin pproximations of elliptic problems, *J. Comput. Appl. Math.*, **226** (2009), 358-369.

[30] R. Verfürth, A Review of A Posteriori Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, New York, Stuttgart, 1996.

[31] M.F. Wheeler, An elliptic collocation-finite element method with interior penalties, *SIAM J. Numer. Anal.*, **15** (1978), 152-161.

[32] B.I. Wohlmuth and R.H.W. Hoppe, A comparison of a posteriori error estimators for mixed finite element discretizations by Raviart-Thomas elements, *Math. Comp.*, **68** (1999), 1347-1378.

[33] J. Yang and Y. Chen, A unified a posteriori error analysis for discontinuous Galerkin approximations of reactive transport equations, *J. Comput. Math.*, **24** (2006), 425-434.